Lecture 2: Hardy spaces in Widom domains

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Introduction

- Reflectionless Jacobi matrices. Remling’s Theorem.

Widom domains and DCT.

Main theorem

- From reflectionless Jacobi matrix to spectral data: $J(E) \rightarrow D(E)$.
- The Abel map: $D(E) \rightarrow \Gamma^*$.
- Functional model: $\Gamma^* \rightarrow J(E)$.
- Theorem

CMV matrices and canonical systems
Remling’s theorem deals with the asymptotic behavior of coefficients sequences of Jacobi matrices having absolutely continuous spectrum. Let $J : l^2 \to l^2$ be the bounded operator generated by a Jacobi matrix

$$Je_n = p_{n+1}e_{n+1} + q_ne_n + p_{n-1}e_{n-1},$$

where $\{e_n\}$ is the standard basis in $l^2$. We will consider also operators acting on the half–axis $J_+ : l^2_+ \to l^2_+$. We say that $J$ is reflectionless on a set $A \subset \sigma(J), |A| > 0$, if

$$R_{n,n}(x + i0) := \langle (J - (x + i0))^{-1}e_n, e_n \rangle \in i\mathbb{R}, \quad \text{a.e. } x \in A. \tag{1}$$

For a given $J_+ = J_+ (\{p_n^+, q_n^+\})$ we say that $J$ belongs to the limit set $\Lambda(J_+)$ if

$$p_n = \lim_{m_k \to +\infty} p_{n+m_k}^+, \quad q_n = \lim_{m_k \to +\infty} q_{n+m_k}^+ \tag{2}$$

for all integers $n \in \mathbb{Z}$ and an unbounded positive sequence $\{m_k\}$.

**Theorem (R)**

Let $E$ be the absolutely continuous spectrum of $J_+, |E| > 0$, and $J \in \Lambda(J_+)$. Then $J$ is reflectionless on the set $E$. 

Remark. In the simplest case when $E$ is an interval, which we can normalize to $[-2,2]$ Theorem (R) is the seminal Rakhmanov’s theorem. Indeed, assume

$$
\sigma_{a.c.}'(x) \neq 0 \quad \text{for almost all } x \in [-2, 2]. \quad (3)
$$

Orthonormal polynomials $P_n(x)$ with respect to this measure satisfy the three term recurrence relation

$$
xP_n(x) = p_{n+1}^+ P_{n+1}(x) + q_n^+ P_n(x) + p_n^+ P_{n-1}(x). \quad (4)
$$

The recurrence coefficients form $J_+$ with the spectrum $[-2,2]$. Due to Rakhmanov’s condition (3), by Theorem (R) every limit point $J \in \Lambda(J_+)$ is reflectionless on this interval. But there exists a unique reflectionless matrix with the spectrum $[-2,2]$, which is the Chebyshev matrix with constant coefficients. That is, by (2),

$$
\lim_{n \to +\infty} p_n^+ = 1, \quad \lim_{n \to +\infty} q_n^+ = 0.
$$
Let $S$ be the shift in $l^2(\mathbb{Z})$. $J$ is said to be \textbf{almost periodic} (a.p.) if the set of operators $\{S^{-n}JS^n\}_{n \in \mathbb{Z}}$ is a precompact in the operator topology.

Let $G$ be a compact Abelian group, $p(\alpha)$ and $q(\alpha)$ be continuous functions on $G$, $p(\alpha) > 0$. For $\tau \in G$, define $T\alpha = \tau \alpha$. Then $J(\alpha)$

$$
(J(\alpha)x)_n = p_n(\alpha)x_{n-1} + q_n(\alpha)x_n + p_{n+1}(\alpha)x_{n+1},
$$

where $p_n(\alpha) = p(T^n\alpha)$ and $q_n(\alpha) = q(T^n\alpha)$, is a.p. Moreover, an a.p. $J$ can be obtained by this construction for some $G$, $p(\alpha)$, $q(\alpha)$, $\tau$.

**Claim.** By Remling's theorem an a.p. $J$ is reflectionless on $E = \sigma_{a.c.}(J)$ (in fact, this is Kotani's theorem).
Let $S$ be the shift in $l^2(\mathbb{Z})$. $J$ is said to be **almost periodic** (a.p.) if the set of operators $\{S^{-n}JS^n\}_{n \in \mathbb{Z}}$ is a precompact in the operator topology.

Let $G$ be a compact Abelian group, $p(\alpha)$ and $q(\alpha)$ be continuous functions on $G$, $p(\alpha) > 0$. For $\tau \in G$, define $T\alpha = \tau\alpha$. Then $J(\alpha)$

$$
(J(\alpha)x)_n = p_n(\alpha)x_{n-1} + q_n(\alpha)x_n + p_{n+1}(\alpha)x_{n+1},
$$

(5)

where $p_n(\alpha) = p(T^n\alpha)$ and $q_n(\alpha) = q(T^n\alpha)$, is a.p. Moreover, an a.p. $J$ can be obtained by this construction for some $G$, $p(\alpha)$, $q(\alpha)$, $\tau$.

**Claim.** By Remling’s theorem an a.p. $J$ is reflectionless on $E = \sigma_{a.c.}(J)$ (in fact, this is Kotani’s theorem).

**Remark.** Periodic case is described by $JS^N = S^N J$.

Note that the structure of $J(\alpha)$ is described by

$$
J(\alpha)S = SJ(T\alpha).
$$

(6)

This indicates strongly that one can also associate with the family $\{J(\alpha)\}$ a pair of commuting operators.
Let $L^2_{d\chi}(l^2(\mathbb{Z}))$ be the space of $l^2(\mathbb{Z})$–valued vector functions, $x(\alpha) \in l^2(\mathbb{Z})$, with the norm

$$\|x\|^2 = \int_G \|x(\alpha)\|^2 d\chi,$$

where $d\chi$ is the Haar measure on $G$. Define

$$(\hat{J}x)(\alpha) = J(\alpha)x(\alpha), \quad (\hat{S}x)(\alpha) = Sx(T\alpha), \quad x \in L^2_{d\chi}(l^2(\mathbb{Z}))$$

Then the commutative relation (6) implies

$$(\hat{J}\hat{S}x)(\alpha) = J(\alpha)Sx(T\alpha) = SJ(T\alpha)x(T\alpha) = (\hat{S}\hat{J}x)(\alpha).$$

Further, $\hat{S}$ is a unitary operator and $\hat{J}$ is selfadjoint. The space $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ is an invariant subspace for $\hat{S}$. It is not invariant for $\hat{J}$ but it does for the product $\hat{J}\hat{S}$. Let us put

$$\hat{S}_+ = \hat{S}|L^2_{d\chi}(l^2(\mathbb{Z}_+)), \quad (\hat{J}\hat{S})_+ = \hat{J}\hat{S}|L^2_{d\chi}(l^2(\mathbb{Z}_+)).$$

We are interested in a functional model, where $\hat{S}_+$ and $(\hat{J}\hat{S})_+$ became operators multiplication by functions in a functional space on an appropriate Riemann surface.
We say that a pair of commuting operators $\hat{S}_+$ and $(\hat{J}\hat{S})_+$ has a (local) functional model if there is a unitary embedding of the space $L^2_{d\chi}(l^2(\mathbb{Z}_+))$ in a space $X_O$, consisting of holomorphic in some domain $O$ functions $F(\zeta)$, $\zeta \in O$, with a reproducing kernel ($F \mapsto F(\zeta_0)$, $\zeta_0 \in O$, is a bounded functional in $X_O$) and under this embedding the operators became a pair of operators multiplication by holomorphic functions, say

$$\hat{S}_+x \mapsto b(\zeta)F(\zeta), \quad (\hat{J}\hat{S})_+x \mapsto (zb)(\zeta)F(\zeta).$$

In fact, such assumptions imply quite strong consequences. Let $k_\zeta$ be the reproducing kernel in $X_O$ and let $\hat{k}_\zeta$ be its preimage in $L^2_{d\chi}(l^2(\mathbb{Z}_+))$. Then

$$\langle \hat{S}_+^* \hat{k}_\zeta, x \rangle = \langle \hat{k}_\zeta, \hat{S}_+x \rangle = \langle k_\zeta, bF \rangle = \overline{b(\zeta)F(\zeta)} = \langle \overline{b(\zeta)k_\zeta}, F \rangle = \langle \overline{b(\zeta)\hat{k}_\zeta}, x \rangle.$$

That is $\hat{k}_\zeta$ is an eigenvector of $\hat{S}_+^*$ with the eigenvalue $\overline{b(\zeta)}$. In the same way, $\hat{k}_\zeta$ is an eigenvector of $(\hat{J}\hat{S})_+^*$ with the eigenvalue $\overline{(zb)(\zeta)}$. 
Thus, if a functional model exists then the spectral problem

\[
\begin{align*}
\hat{S}_+^* \hat{k}_\zeta &= b(\zeta) \hat{k}_\zeta, \\
(\hat{J}\hat{S})_+^* \hat{k}_\zeta &= (zb)(\zeta) \hat{k}_\zeta,
\end{align*}
\]

has a solution \( \hat{k}_\zeta \) with an anti–holomorphic dependence of \( \zeta \). Vice–versa, if (7) has a solution of this kind, and the system of eigenvectors \( \{ \hat{k}_\zeta \}_{\zeta \in \mathcal{O}} \) is complete in \( L^2_{d\chi}(l^2(\mathbb{Z}_+)) \) then we put

\[
F(\zeta) = \langle x, \hat{k}_\zeta \rangle, \quad \|F\|^2 = \|x\|^2,
\]

and this provide a local functional model for the pair \( \hat{S}_+, (\hat{J}\hat{S})_+ \).

At least, under some assumptions on the spectrum of an almost periodic Jacobi matrix we can present a global functional model of this kind, where \( b \) is the symbol of the shift operator and \( z \) is the symbol of \( J \).
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CMV matrices and canonical systems
We use the standard terminology and notion of the theory of functions of bounded characteristics in $\mathbb{D}$. $H^p$ denotes the Hardy space. $f$ is of Smirnov class if $f = f_1/f_2$, where $f_1, f_2 \in H^\infty$ and $f_2$ is outer.

$\Gamma$ is a discrete subgroup of $SU(1,1)$

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \quad \gamma_{11} = \overline{\gamma_{22}}, \quad \gamma_{12} = \overline{\gamma_{21}}, \quad \det \gamma = 1,$$

$$\gamma(\zeta) = (\gamma_{11}\zeta + \gamma_{12})/(\gamma_{21}\zeta + \gamma_{22}).$$

**Uniformization theorem.** If $E = [b_0, a_0] \cup \bigcup_{j \geq 1} (a_j, b_j)$ then $\overline{\mathbb{C}} \setminus E \sim \mathbb{D}/\Gamma$.

\[
\{\text{merom. f. } f \text{ in } \mathbb{D}, \text{ s.t. } f(\gamma(\zeta)) = f(\zeta)\} \equiv \{\text{merom. f. } F \text{ in } \overline{\mathbb{C}} \setminus E\}
\]
The dual group \( \alpha \in \Gamma^* \iff \alpha : \Gamma \to \mathbb{T}, \alpha(\gamma_1 \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2) \).

**Definition.** \( H^\infty(\alpha, \Gamma) = H^\infty(\alpha) = \{ f \in H^\infty : f \circ \gamma = \alpha(\gamma)f \} \).

\( \Omega = \overline{\mathbb{C}} \setminus E \) is of Widom type if \( H^\infty(\alpha) \) are non trivial for all \( \alpha \in \Gamma^* \).

Widom condition: \( \sum h_k < \infty \).

**Complex Green function:**
\[
B^\infty(z(\zeta)) = e^{iw} = \zeta \prod_{\gamma \in \Gamma, \gamma \neq id} \frac{|\gamma(0)|}{\gamma(0)} \frac{\gamma(0) - \zeta}{1 - \zeta \gamma(0)}
\]

**Definition.** DCT holds if for all \( F \) of Smirnov class in \( \Omega \)
\[
\frac{1}{2\pi i} \oint_{\partial \Omega} \frac{F(z)dz}{z - z_0} = F(z_0), \text{ as soon as } \int_E |F(x)|dx < \infty.
\]

\( E^1_\Omega = \{ F(z) \text{ is of Smirnov class and } \int_E |F(x)|dx < \infty \} \).
Definition. $A^2_1(\alpha)$ is formed by Smirnov class functions such that
\[
\frac{f(\gamma(\zeta))}{\gamma_{21}\zeta + \gamma_{22}} = \alpha(\gamma)f, \text{ i.e. } |f(\zeta)|^2|d\zeta| \text{ is } \Gamma - \text{invariant},
\]

\[
\int_{E} |f|^2 \, dm < \infty : m(\bigcup_{\gamma \in \Gamma} \gamma(E)) = m(\mathbb{T}), \gamma(E) \cap E = \emptyset, \gamma \neq \text{id}.
\]

Claim. For Widom domains all three mentioned possible realization

- $H^2(\alpha) = \{f \in H^2 : f \circ \gamma = \alpha(\gamma)f\}$
- $E^2(\alpha) = \{\text{multivalued } F(z) \text{ of Smirnov class in } \Omega : \int_E |F(x)|^2 \, dx < \infty, F \circ \gamma = \alpha(\gamma)F, \gamma \in \pi_1(\Omega)\}$
- $A^2_1(\alpha) = \left\{ \frac{f(\gamma(\zeta))}{\gamma_{21}\zeta + \gamma_{22}} = \alpha(\gamma)f, \int_{E} |f|^2 \, dm < \infty \right\}$

are equivalent:

\[
k_{A^2_1(\alpha)}(\zeta, \zeta) = k_{H^2(\alpha\tau_1)}(\zeta, \zeta)\phi_1(\zeta)\overline{\phi_1(\zeta)} = K_{E^2(\alpha\tau_2)}(z(\zeta), z(\zeta))\phi_2(\zeta)\overline{\phi_2(\zeta)}.
\]

In fact,

\[
\left\{ \begin{array}{ll}
\hat{S}_+^* \hat{k}_\zeta = \overline{b(\zeta)} \hat{k}_\zeta \\
(\hat{J}\hat{S})_+^* \hat{k}_\zeta = (zb)(\zeta) \hat{k}_\zeta,
\end{array} \right.
\]

defines an (anti)-holomorphic vector bundle.
Theorem

Let $L^2_E$ be the space of square-integrable functions on $E$ with respect to $dm$. If $g(\zeta) = \zeta f(\zeta) \in L^2_E \ominus A^2_1(\alpha)$ then $f \in A^2_1(\alpha^{-1})$.

- Moreover, DCT holds if and only if

$$L^2_E = \zeta A^2_1(\alpha^{-1}) \oplus A^2_1(\alpha) \quad \forall \alpha \in \Gamma^*.$$ 

Also DCT is equivalent to the conditions

- The function $k^\alpha(0)$ is continuous on $\Gamma^*$.
- The following character-automorphic counterpart of Beurling’s Theorem holds true: Every invariant subspace $M \subset A^2_1(\alpha)$ (here invariance means that $wM \subset M$ for all $w \in H^\infty(\Gamma)$) is of the form

$$M = \Theta A^2_1(\alpha \tau^{-1})$$

where $s$ is a character-automorphic inner function and $\sigma$ denotes its character, $\Theta \circ \gamma = \tau(\gamma) \Theta$. 

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CMV matrices and canonical systems
\[ J(E) \rightarrow D(E) \]

By \( J(E) \) we denote the set of reflectionless Jacobi matrices \( J \) with the spectrum on \( E \). The resolvent functions possess the representation

\[
R(z) = R_{n,n}(z) = \langle (J - z)^{-1} e_n, e_n \rangle = \int_E \frac{d\sigma(x)}{x - z}.
\]

Reflectionless means

\[
\text{arg } R(x) = \begin{cases} 
\frac{\pi}{2}, & x \in E \\
0, & x_j < x < b_j \\
\pi, & a_j < x < x_j
\end{cases}
\]

Therefore

\[
R(z) = R(z, \{x_j\}) = -\frac{1}{\sqrt{(z - a_0)(z - b_0)}} \prod_{j \geq 1} \frac{z - x_j}{\sqrt{(z - a_j)(z - b_j)}}.
\]
By $D(E)$ we denote the set of so-called divisors (spectral data)

$$D = \{(x_j, \epsilon_j) : x_j \in [a_j, b_j], \epsilon_j = \pm 1\}, (a_j, 1) \equiv (a_j, -1), (b_j, 1) \equiv (b_j, -1)$$

$J(E) \rightarrow D(E)$ is defined as follows. For $J \in J(E)$ the res. f. $R_{00}$ possesses the special representation and this defines $\{x_j\}$. To define $\epsilon_j$ we write

$$J = \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} + p_0 e_{-1} \langle \cdot, e_0 \rangle + p_0 e_0 \langle \cdot, e_{-1} \rangle.$$ 

This representation generates the identity

$$-\frac{1}{R_{00}(z)} = -\frac{p_0^2}{r_{-}(z)} + r_+(z), \quad r_{\pm}(z) = \langle (J_{\pm} - z)^{-1} e_{\pm 1-1}, e_{\pm 1-1} \rangle \quad (8)$$

Claim: $x_j \in (a_j, b_j)$ is a pole of only one of the two ($\pm$) functions.

We set $\epsilon_j = 1$ if $x_j$ is a pole of $r_+$ and $\epsilon_j = -1$ in the opposite case.

**Remark.** If $-1/R_{00}$ has no singular component on $E$ then (8) defines $r_{\pm}$, and therefore $J$, uniquely. That is, $J(E) \rightarrow D(E)$ is one-to-one.
Let $c_k$ be the critical points of the complex Green function $B(z) = B_\infty(z)$, note $c_k \in (a_k, b_k)$. The Widom condition guaranties that the following products converge

$$K^D(\zeta) = \sqrt{\frac{dB(z(\zeta))}{d\zeta}} \prod_{j \geq 1} \frac{z(\zeta) - x_j}{(z(\zeta) - c_j)B_{x_j}(z(\zeta))} \prod_{j \geq 1} B_{x_j}(z(\zeta))^{1+\epsilon_j} \quad (9)$$

The Abel map $D \mapsto \alpha$ is defined by

$$\frac{K^D(\gamma(\zeta))}{\gamma_{21}\zeta + \gamma_{22}} = \alpha^D(\gamma)K^D(\zeta).$$

**Remark.** The reproducing kernel $k^\alpha$ is of the form (9) with an appropriate $D$. Every function $K^D$ is the reproducing kernel in the corresponding $A^2_1(\alpha)$ if and only if DCT holds.
The system of functions $e_n(\zeta) = B^n(z(\zeta)) \frac{k^{\alpha \mu^{-n}}}{\|k^{\alpha \mu^{-n}}\|}$ forms an orthonormal basis in $A^2_1(\alpha)$ for $n \in \mathbb{Z}_+$ and in the whole space $L^2_E$ for $n \in \mathbb{Z}$.
Theorem

Denote by $\mu$ the character of $B$, $B \circ \gamma = \mu(\gamma)B$. The system of functions

$$e_n(\zeta) = B^n(z(\zeta)) \frac{k^{\alpha \mu - n}}{\|k^{\alpha \mu - n}\|}$$

forms an orthonormal basis in $A^2_1(\alpha)$ for $n \in \mathbb{Z}_+$ and in the whole space $L^2_E$ for $n \in \mathbb{Z}$. The multiplication operator by $z(\zeta)$ with respect to this basis is a Jacobi matrix $J(\alpha) \in J(E)$. Moreover

$$p_n(\alpha) = \mathcal{P}(\alpha \mu^{-n}), \quad q_n(\alpha) = \mathcal{Q}(\alpha \mu^{-n}), \quad \text{where} \quad (10)$$

$$\mathcal{P}(\alpha)^2 = (zB)(0) \frac{k^{\alpha}(0)}{k^{\alpha \mu}(0)}, \quad B'(0)\mathcal{Q}(\alpha) = (zB)'(0) + \left(\log \frac{k^{\alpha}}{k^{\alpha \mu}}\right)'(0).$$
\[ \Gamma^* \to J(E) \]

**Theorem**

Denote by \( \mu \) the character of \( B \), \( B \circ \gamma = \mu(\gamma)B \). The system of functions

\[ e_n(\zeta) = B^n(z(\zeta)) \frac{k^{\alpha\mu^{-n}}}{\|k^{\alpha\mu^{-n}}\|} \]

forms an orthonormal basis in \( A_1^2(\alpha) \) for \( n \in \mathbb{Z}_+ \) and in the whole space \( L^2_{\mathbb{E}} \) for \( n \in \mathbb{Z} \). The multiplication operator by \( z(\zeta) \) with respect to this basis is a Jacobi matrix \( J(\alpha) \in J(E) \). Moreover

\[ p_n(\alpha) = P(\alpha\mu^{-n}), \quad q_n(\alpha) = Q(\alpha\mu^{-n}), \text{ where} \]

\[ P(\alpha)^2 = (zB)(0) \frac{k^\alpha(0)}{k^{\alpha\mu}(0)}, \quad B'(0)Q(\alpha) = (zB)'(0) + \left( \log \frac{k^\alpha}{k^{\alpha\mu}} \right)'(0). \]

**“Proof”**. \( A_1^2(\alpha) = \{k^\alpha\} \oplus \{f \in A_1^2(\alpha) : f(0) = 0\} = \{k^\alpha\} \oplus BA_1^2(\alpha\mu^{-1}). \)
Theorem (Main)

Let $E$ be such that $\Omega = \overline{\mathbb{C}} \setminus E$ is of Widom type with DCT. Then every $J \in J(E)$ is almost periodic.

Proof.

$J \to D \to \alpha$. $P(\alpha)$ and $Q(\alpha)$ are continuous functions on the compact Abelian group $\Gamma^*$.

Example

A compact $E$ is homogeneous if there exists $\eta > 0$ such that

$$|E \cap (x - \delta, x + \delta)| \geq \eta \delta$$

for all $x \in E$ and $\delta \in (0,1)$. If $E$ is homogeneous then $\Omega$ is of Widom type with DCT. For instance all Cantor sets of positive length (Cantor type construction with a non-constant ratio) are homogeneous.
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CMV matrices and canonical systems
CMV matrices

For a given sequence \( \{a_k\}_{k=-\infty}^{\infty}, a_k \in \mathbb{D} \), define \( A = A(\{a_k\}) = A_0 A_1 \)

\[
A_0 = \begin{bmatrix}
    \cdots & A_{-2} \\
    & A_0 \\
    & \cdots
\end{bmatrix}, \quad A_1 = S \begin{bmatrix}
    \cdots & A_{-1} \\
    & A_1 \\
    & \cdots
\end{bmatrix} S^{-1}
\]

\( A_j = \begin{bmatrix}
    \bar{a}_j & \rho_j \\
    \rho_j & -a_j
\end{bmatrix}, \quad \rho_j = \sqrt{1 - |a_j|}, \) and operators act in \( l_2^+ \oplus l_2^- \).

The functional model for a.p. CMV(\(E\)):
multiplication by \( v = \frac{z - z_0}{\bar{z} - \bar{z}_0} = \frac{\zeta}{b_\alpha(\zeta)} \) in \( L_2^E \) w.r.t.

\[
e_n(\alpha, e^{ic}) = \begin{cases}
    b^m b^m_\alpha \frac{k_{\alpha \mu}^{-n}}{\|k_{\alpha \mu}^{-n}\|} e^{ic}, & n = 2m \\
    b^{m+1} b^m_\alpha \frac{k_{\alpha \mu}^{-n}}{\|k_{\alpha \mu}^{-n}\|}, & n = 2m + 1
\end{cases}
\]
Martin function and canonical systems

Inner function with a unique singularity point on $\partial \Omega$. For $x_0 \in E$

$$\Theta(z, x_0) = \lim_{z_0 \to x_0} i \frac{\log B(z, z_0) B(z, \bar{z}_0)}{\log |B(\infty, z_0)|^2}.$$ 

Multiplication by $z$ with respect to the chain

$$\{ e^{it\Theta(z(\zeta))} A_1^2(\alpha \delta_t^{-1}) \}_{t=-\infty}^\infty, \quad e^{it\Theta} \circ \gamma = \delta_t(\gamma) e^{it\Theta},$$

leads to a canonical system.
References


Theorem

Let $\Omega$ be of Widom type with DCT. The set $J(E)$ can be represented in the form (5) with $G = \Gamma^*$, $T_\alpha = \mu^{-1} \alpha$ and $P(\alpha)$ and $Q(\alpha)$ of the form (10).

The operators $\hat{S}_+$ and $(\hat{S} \hat{J})_+$ are unitary equivalent to multiplication by $B$ and $zB$ in $A^2_1(\Gamma')$ respectively. This unitary map is given by the formula

$$\sum_{\{\gamma\} \in \Gamma / \Gamma'} f|[\gamma] \alpha^{-1}(\gamma) = \sum_{n \in \mathbb{Z}_+} x_n(\alpha) e_n(\alpha)$$

where $f \in A^2_1(\Gamma')$ and the vector function $x(\alpha) = \{x_n(\alpha)\}$ belongs to $L^2_{d\chi}(l^2_+)$. 