Special conformal mappings in Approximation Theory

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Two classical problems

**Polynomials of least deviation from zero.** Let $E' \subset \mathbb{R}$, and $P_n$ a polynomial with minimal sup-norm $L_n = \|P_n\|_{E'}$ among all monic polynomials of degree $n < \text{card } E'$. Find $P_n$.

**Spectra of periodic Jacobi matrices.** A periodic Jacobi matrix $J$ defines a bounded self-adjoint operator in $\ell^2$

$$Je_k = p_k e_{k-1} + q_k e_k + p_{k+1} e_{k+1}, \quad p_k > 0, \; q_k \in \mathbb{R}.$$ 

Find spectrum of $J$.

**Solutions.** Let $\Pi_n = \Pi(h_1, h_2, \ldots, h_{n-1})$ be a region obtained from the half-strip by removing vertical intervals $\{-\pi k + it : 0 \leq t \leq h_k\}$.

There exists $\Pi_n$ s.t.

$$P_n(z) = L_n \cos \Theta_n(z)$$

$$\sigma(J) = \Theta_n^{-1}([-\pi n, 0])$$
Examples: Extremal Polynomials & Entire Functions (EF)

Hayman’s Problem (A. Eremenko). For \( A = \cosh h > 1 \) find

\[
L(z) = \sup \left\{ |F(z)| : F \text{ is EFET} \leq 1, \ |F(x)| \leq \begin{cases} 
1, & x < 0 \\
A, & x > 0 
\end{cases} \right\}
\]

\[
\Theta(0) = 0-, \ \Theta(z) = z + \ldots, \quad L(x) = \cos \Theta(x), \quad 0 \leq x \leq a := \Theta^{-1}(ih).
\]
Examples: Green’s & Martin’s Functions

**Proposition 1.** Let \( \theta = \Theta/n \). Then \( \text{Im} \theta(z) = G(z, \infty, \bar{C} \setminus E) \), \( \text{Im} z > 0 \).

**Proposition 2.** A system of intervals \( E = [b_0, a_0] \setminus \cup (a_j, b_j) \) represents the spectrum of a periodic \( J \) iff \( \omega([b_k, a_{k+1}], \infty, \bar{C} \setminus E) \in \mathbb{Q} \).

**Proposition 3.** A compact \( E \) is regular in the sense of potential theory iff \( \Pi \) is obtained from the strip by making countably many cuts, such that \( h_k \to 0, k \to \infty \).

**Example.** \( E := \theta^{-1}([-\pi, 0]) \) corresponds to the Julia set of the quadratic polynomial \( T(z) = z^2 - \lambda \), \( \theta^{-1}(-\pi + 2ih_0) = -\lambda < -2 \)
\( \theta^{-1}(-\pi) = -\xi, \ \theta^{-1}(0) = \xi = T(\xi) \).
Examples: Green’s & Martin’s Functions

Proposition 1. Let $\theta = \Theta/n$. Then $\text{Im}\theta(z) = G(z, \infty, \mathbb{C} \setminus E)$, $\text{Im}z > 0$.

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Proposition 4. A domain $\mathbb{C} \setminus E$ is of Parreau-Widom type iff $\sum h_k < \infty$.

Def. $\text{PW} = H^\infty(\alpha)$ contains a non-constant function for all $\alpha \in \pi^*_1(\mathbb{C} \setminus E)$. 
Generalized Chebyshev Polynomials

\[ P = \cos \Theta, \quad \Theta : \mathbb{H} \to \Pi(h_1, h_2, \ldots, h_{n-1}) \]


- Polynomials with real ±1 values.
- Collinear to polynomials with real and simple zeros, \( P = \pm L \cos \Theta \).

**Theorem**

*For every finite sequence \( c_1, c_2, \ldots, c_{n-1} \) with the property \((-1)^k c_k \geq 1\) there exists \( P(z) = Cz^n + \ldots, \ C > 0, \) for which this (ordered!) sequence is the sequence of its (all) critical values. It is defined by this sequence up to a change \( z \mapsto az + b \) of independent variable, \( a > 0, \ b \in \mathbb{R} \).*  

*Proof.* \( c_k = P(x_k), \ P'(x_k) = 0, \ x_{k+1} < x_k. \ c_k = (-1)^k \cosh h_k. \)
Comb representation of LP (Laguerre-Pólya) polynomials

**Def.** LP-class is formed by real polynomials with all zeros real. We normalized such polynomial on a positive leading term.

**Theorem**

\[ P \in \text{LP} \iff P = e^\phi, \text{ where } \phi : \mathbb{H} \to \Omega = \Omega(h_1, h_2, \ldots, h_{n-1}) \text{ is a conformal map onto a V-comb [Vinberg, 1989], } h_k \in [-\infty, \infty). \]

**Corollary.** LP-polynomial is uniquely defined by its sequence of critical values, \( c_k = (-1)^k e^{h_k} \), up to \( z \mapsto az + b \).
Example: uniform Jacobi polynomials

Let $\alpha, \beta \geq 0$ and let $J_n(x; \alpha, \beta) = x^n + \ldots$ denote the monic extremal polynomial on $[0, 1]$ with respect to the weight function $x^\alpha (1 - x)^\beta$.

**Lemma.** For nonnegative $\alpha, \beta$ and an integer $n$

$$x^\alpha (1 - x)^\beta J_n(x) = L_n e^\phi, \quad \phi : \mathbb{H} \to \Omega, \quad h_k = 0.$$
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**Theorem (Moale-Peherstorfer)**

A complex Chebyshev polynomial with the leading term $z_1^{k_1} \ldots z_d^{k_d} \bar{z}_1^{\ell_1}$ in the ball $\sum_{j=1}^d |z_j|^2 \leq 1$ is of the form

$$z_1^{k_1 - \ell_1} z_2^{k_2} \ldots z_d^{k_d} J_{\ell_1} \left( |z_1|^2; \frac{k_1 - \ell_1}{2}, \frac{k_2 + \ldots + k_d}{2} \right), \; k_1 \geq \ell_1.$$


Problem (W. Hayman, H. Stahl). For \( A > 1 \) and \( B > 1 \), find the polynomial \( P_n(x) \) of least deviation from \( \text{sgn}(x) \) on the union \( [-A, -1] \cup [1, B] \) and the asymptotics for the error \( L_n = L_n(A, B) \).


Asymptotics for \( L_n \) is of the form

\[
L_n = \left( c + o(1) \right) n^{1/2} e^{-n \eta} \left| \vartheta_0 \left( \frac{1}{2} \left( \{ n \omega_1 + \omega_2 \} - \omega_2 \right) \tau \right) \right|,
\]

where \( \{ x \} \) denotes the fractional part of \( x \), the constants \( \tau, c, \eta, \omega_1, \omega_2 \) are given explicitly by means of elliptic integrals depending on \( A \) and \( B \), and

\[
\vartheta_0(v|\tau) = 1 - 2h \cos 2\pi v + 2h 4 \cos 4\pi v - 2h 9 \cos 6\pi v + \ldots,
\]

\( h = e^{\pi i \tau} \).


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Theorem (A. Eremenko, P. Yuditskii, J. Anal. Math., to appear) Asymptotics for $L_n$ is of the form $L_n = (c + o(1))n^{-1/2}e^{-\eta|\theta_0(\frac{1}{2}(\{n\omega_1 + \omega_2\} - \omega_2)|\tau)}$, where $\{x\}$ denotes the fractional part of $x$, the constants $\tau, c, \eta, \omega_1, \omega_2$ are given explicitly by means of elliptic integrals depending on $A$ and $B$, and \( \theta_0(v|\tau) = 1 - 2h\cos 2\pi v + 2h^2\cos 4\pi v - 2h^3\cos 6\pi v + \ldots, \) $h = e^{\pi i\tau}$. 

P. Yuditskii (JKU)
• Doron Lubinsky, *Best Approximating Entire Functions of Exponential Type*, Third International Conference on Complex Analysis and Dynamical Systems, January 2–6, 2006, in Galilee, Israel.


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\]
Shape of the extremal function and MacLane’s Theorem

$sYgn(z; a)$: extremal EFET $\leq 1$ for $\text{sgn}(x)$ on the set $\mathbb{R} \setminus (-a, a)$.

EF related to the Hayman problem (in MO), also has only 4 critical values: $\pm 1, \pm A$. 
Shape of the extremal function and MacLane’s Theorem

**Theorem** (MacLane). For every up-down sequence \( \{ c_k \} \) there exists \( f \) with the only real critical values for which it is the critical sequence. Any two functions are related by \( f_1(z) = f_2(\alpha z + \beta) \).

**Remarks.**
1. \( f \in M \) (MacLane) \( \Rightarrow f' \in LP \). Thus the theorem shows that critical values of integrals of LP-functions can be arbitrary prescribed, subject to the evident restriction \( (c_{k+1} - c_k)(c_k - c_{k-1}) \leq 0 \).
2. Since \( f \in LP \Rightarrow f' \in LP \). That is, \( LP \subset M \). Recall \( MO \subset LP \).
3. \( sYgn(z) \) one of the simplest function which belongs to \( M \), but not \( LP \). Due to MacLane’s theorem it is uniquely restored by its shape, \( L = L(a) \).
sYgn(z) as a special function

\[ sYgn(z, a) = S(\sqrt{z^2 + a^2}) \]

Extremal polynomial via conformal mapping \( P_n(z) = S(\phi_n(z)) \)
Step 1
Step 2
Step 3
From an interval and a point to two symmetric intervals
Leading term in asymptotics

**Renormalized domain**

$$\alpha = \lim_{n \to \infty} \frac{\text{#alternance points on } [-A, -1]}{n}$$

**Limit domain**

$$\eta = \lim_{n \to \infty} \frac{a_n}{n}, \quad D_* = \lim_{k \to \infty} d_{nk}$$
Almost periodic "finite zone" multi-diagonal matrices: parametrization of spectral curves by branching divisors

\[ \sigma_{c_1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \ldots; \sigma_{c_1} \cdots \sigma_{c_\ell} = \sigma_{\infty}^{-1} \]
Almost periodic "finite zone" multi-diagonal matrices: parametrization of spectral curves by branching divisors

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Almost periodic "finite zone" multi-diagonal matrices

\[ z(b_{\infty_1} \ldots b_{\infty_m}) - \text{holo}, \quad n_1 + \cdots + n_m = n. \]

\[ b = (b_{\infty_1}^{n_1} \ldots b_{\infty_m}^{n_m})^{1/n} \sim S, \quad z \sim J \]