Saks spaces

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Contents

1 Saks spaces

Definition 1 *A* **Saks space** *is a triple* $(E, || \, ||, \tau)$*, where*

1. *E is a vector space;*
2. *$|| \, ||$ is a norm and τ eine locally convex topology;*
3. *τ is weaker than $\tau_{|| \, ||}$, but $B_{|| \, ||}$ is τ-complete (and so closed).*

REMARK. Hence $(E, || \, ||)$ is a Banach space.

EXAMPLES.

1. Let S be a $T_{3\frac{1}{2}}$ $k_{\mathbf{R}}$-space, (for example, if S is locally compact or metrisable), $E = C^\infty(S)$, $|| \, || = || \, ||_\infty$, $\tau = \tau_{\mathcal{K}}$ (compact convergence).
2. (The original Example of Saks):

   $E = L^\infty(\mu)$, where μ is a finite measure, $|| \, || = || \, ||_\infty$, $\tau = \tau_{|| \, ||_1}$ (the L^1-Norm).
3. $H^\infty(U) = \{f : U \to \mathbf{C} : f \text{ holomorphic , bounded}\}$, $|| \, || = || \, ||_\infty$, τ is the topology of compact convergence.
4. $E = L(H)$ $|| \, ||$ = the operator norm, τ_w the weak topology, τ_s the strong topology.

Let $(E, || \, ||, \tau)$ be a Saks space. We define a locally convex topology $\gamma(|| \, ||, \tau)$ on E as follows:

1. Let U be absolutely convex. Then U is a γ-neighbourhood of zero $\Leftrightarrow U \cap B$ is a neighbourhood of zero in (B, τ_B).

   alternatively
2. Let $\mathcal{U} = (\mathcal{U}_\backslash)$ be a sequence of τ-neighbourhoods of zero,

   $$\gamma(U) := \bigcup_{n=1}^{\infty} (U_1 \cap B + U_2 \cap 2B + \cdots + U_n \cap nB).$$

   Then the family of all such sets is a basis of γ-neighbourhoods of zero.

Proposition 1 *1. $\tau \subseteq \gamma \subseteq \tau_{|| \, ||}$*

2. γ coincides with τ on $|| \, ||$-bounded sets. Furthermore, γ is the finest locally convex topology with this property.

3. *for suitable topologies* τ, τ_1 *we have:*

$$\gamma(|| \, ||, \tau) = \gamma(|| \, ||, \tau_1) \Leftrightarrow \tau|_B = \tau_1|_B$$

4. $B \subseteq E$ *is* $|| \, ||$*-bounded* $\Leftrightarrow B$ *is* γ*-bounded*

5. (E, γ) *is complete*

6. $B \subseteq E$ *is* γ*-compact* $\Leftrightarrow B$ *is* $|| \, ||$*-bounded and* τ*-compact*

7. $x_n \xrightarrow{\gamma} x \Leftrightarrow \sup ||x_n|| < \infty$ *and* $x_n \xrightarrow{\tau} x$

Duality: Let $(E, || \, ||, \tau)$ be a Saks space. Then E has three dual spaces:

$$E'_\tau \subseteq E'_\gamma \subseteq E'_{|| \, ||}.$$

Then: E'_γ is the norm closure of E'_τ in $E'_{|| \, ||}$.
Examples. $E = (L^\infty, , || \, ||_\infty, \tau_{L^1})$, $E'_\tau = L^\infty$, $E'_\gamma = L^1$.

2 Constructions

Products of Banach spaces. Let E_n be a sequence of Banach spaces. We can form three types of product:

$\prod E_n$ – this is a Fréchet space with the product topology;

$B \prod E_n = \{(x_n) \in \prod E_n : \sup ||x_n||_n < \infty\}$ —a Banach space;

$S \prod E_n = B \prod E_n$ as vector space with the norm $|| \, ||$ and the product topology τ.

Now let $\{\pi_n : E_{n+1} \to E_n\}$ be a countable spectrum of Banach spaces, where $||\pi_n|| \leq 1$.

We have three **projective limits** as closed subspaces of the corresponding products

$$LKR- \underset{\leftarrow}{\rightarrow} \lim E_n = \left\{(x_n) \in \prod E_n : \pi_n(x_{n+1}) = x_n \quad (n \in \mathbf{N})\right\}.$$

This is a locally convex space.

$$B- \underset{\leftarrow}{\rightarrow} \lim E_n = \left\{(x_n) \in B \prod E_n : \bigwedge_{n \in \mathbf{N}} \pi_n(x_{n+1}) = x_n\right\}.$$

This is a Banach space.

$$S- \underset{\leftarrow}{\rightarrow} \lim E_n = \left(B- \underset{\leftarrow}{\rightarrow} \lim E_n\right)$$

as Saks space with the above norm and the product topology as τ.

For example let $E_n = C[-n, n]$ and $\pi_n : E_{n+1} \to E_n$ be the restriction mapping

$LKR- \underset{\leftarrow}{\to} \lim E_n = C(\mathbf{R})$ with the topology of compacte convergence

$B- \underset{\leftarrow}{\to} \lim E_n = (C^b(\mathbf{R}), || \, ||_\infty)$

$S- \underset{\leftarrow}{\to} \lim E_n = (C^b(\mathbf{R}), || \, ||_\infty, \tau_{\mathcal{K}})$.

Proposition 2 *Each Saks space has a representation* $S- \underset{\leftarrow}{\to} \lim E_\alpha$*, where* $\{\pi_{\beta_\alpha} : E_\beta \to E_\alpha\}$ *is a projective spectrum of Banach spaces.*

Examples. Let $E = L(H)$, where H is separable (and so isometric to $\ell^2(\mathbf{N})$).

$$E_n = \{(\xi_1 \dots \xi_n, 0 \dots)\} \cong \mathbf{R}^n$$

Let $\pi_{m,n} : E_m \to E_n$ be the orthogonal projekcion $(n < m)$.

${}_{n,m} : E_n \to E_m$ the natural injection.

$L(E_m)$ is the space of $m \times m$ matrices.

Consider the mapping $\Pi_{m,n} : L(E_m) \to L(E_n)$ $(m < n)$, where

$$T \mapsto \pi_{m,n} \circ T \circ i_{n,m}$$

Then $B- \underset{\leftarrow}{\to} \lim L(E_n) = (L(H), || \, ||)$, $S- \underset{\leftarrow}{\to} \lim L(E_n) = (L(H), || \, ||, \tau_w)$.

As examples of results which are valid for $C(K)$-spaces and have natural generalisations to Saks spaces of the form $C^\infty(S)$, we mention the following:

Proposition 3 *(Stone Weierstraß). Let* $A \subseteq C^\infty(S)$ *be a lattice with* $1 \in A$*, so that* A *separates the points of* S*. Then* A *is* γ*-dense.*

Proposition 4 *(Riesz) The dual space of* $(C^\infty(S), \gamma)$ *is the space* $M^t(S)$ *of bounded Radon measures on* S*.*

Definition 2 *A* **Saks algebra** *is a Saks space with a representation* $S- \underset{\leftarrow}{\to} \lim(E_\alpha, \tau_{\beta_\alpha})$ *as limit of a spectrum of Banach algebras, where* π_{β_α} *is multiplicative. (We shall always assume that our algebras are commutative with unit).*

If $(A, || \, ||, \tau)$ is a Saks algebra, then we define

$$M_\gamma(A) := \{f : A \to \mathbf{C} : f ox multiplicative and \gamma ox continuous\}.$$

$M\gamma(A)$ is thus a subset of the Banach algebra spectrum $M_{|| \, ||}(A)$.

EXAMPLES. For $A = (C^\infty(S), || \, ||, \tau_K)$ $M_\gamma(A) = S$.

We remark briefly that using these concepts one can get a satisfactory extension of the classical Gelfand Naimark theor (which establishes the duality between the class of compact spaces and that of the commutativen B^* algebras with unit) to the class of completely regular spaces.

We have the following generalisation of the representation theorem of Riesz:

Proposition 5 *Let $(E, || \, ||, \tau)$ be a Saks space with $B_{|| \, ||}, \tau$-compact. Then each γ-continuous operator $T : C^\infty(S) \to E$ has a representation $T(f) = \int f d\mu$, where μ is a bounded E-valued measure on $Bo(S)$.*

EXAMPLES. If E is a Banach space, then one can regard E' as a Saks space, namely as $(E', || \, ||, \sigma(E', E))$. This space hat compact unit ball.

Conversely, each Saks space F with compact unit ball has the form $F = (E', || \, ||, \sigma(E', E))$ for a Banach space E. Such spaces can also bve characterised as those Saks spaces with representations $F = S\!-\! \underset{\leftarrow}{\to} \lim F_\alpha$, where the F_α are finite dimensional.

Using this last remark we can prove the above result by using the representation as a projective limit of finite dimensional case to reduce to the classical Riesz representation quoted above.

Corollar 1 *Let $T : C(K) \to E$ be weakly compact, (i.e. $T(B_{C(K)})$ is relatively $\sigma(E, E')$-compact) , where E is a Banach space. Then there exists a Radon measure $\mu : Bo(K) \to E$, so that $T(f) = \int f d\mu$.*

Sketch. We put $B = \overline{(T(B_{C(K)}))}$ and consider the Saks space

$$(E_B, || \, ||_B, \sigma(E, E')).$$

This is a Saks space with compact unit ball. In this way one obtains a representing measure which is weakly Radon. One then applies a theorem of Pettis-type.

■

Proposition 6 *(Orlicz-Pettis). Let (x_n) be a sequence in a Banach space, so that $\sum_{i=1}^\infty \epsilon_i x_i$ is weakly convergent for each $(\epsilon_i) \in \{-1, 1\}^{\mathbf{N}}$. Then $\sum x_n$ converges in the norm.*

We now prove

Theorem 1 Eberlein-Smulian theorem. *Let $B \subseteq E$ be bounded. Then the following statements are equivalent*

1. *B ist weakly compact;*

2. *B is weakly σ-compact;*

3. *B is weakly sequentially compact.*

For this we require some notions from the theory of integration for vector-valued functions.

Definition 3 *Let $(\Omega, \mathcal{A})$ be a set with a σ-algebra, E a Banach space. $f : \Omega \to E$ is a* **measurable step function***, if*

$$f = \sum_{k=1}^{n} l_k \chi_{A_k},$$

where the A_k are measurable. f is **measurable***, $\Leftrightarrow f = \underset{n}{\rightarrow} \infty \to \lim f_n$ ((f_n) a sequence of measurable step functions).*

This definition is equivalent to the fact that $f(\Omega)$ is separable and $f^{-1}(U) \in \mathcal{A}$ (U open). Then the $\mathbf{R}$-valued function $\|f\|$ is measurable.

Definition 4 *Let $(\Omega, \mathcal{A}, \mu)$ be a W-Maß. $f : \Omega \to E$ is* **Bochner-integrable***: $\Leftrightarrow f$ is measurable and $\int ||f|| d\mu < \infty$.*

This is equivalent to the fact that f is the pointwise limit of a sequence f_n of step functions so that $\lim \int f_n d\mu$ exists.

In particular: if f is bounded and measurable $\Rightarrow f$ ist Bochner-integrable.

Proposition 7 *Let f be Bochner integrable. Then $\{\int_A f d\mu : A \in \mathcal{A}\}$ is relatively compact in E.*

Proof. Take $\epsilon > 0$ and chose a step function g with $\int ||f - g|| d\mu < \epsilon/2$. $\{\int_A g d\mu : A \in \mathcal{A}\}$ is finitely dimensional and bounded (since $g(\Omega)$ is finite dimensional). Then $\{\int_A f d\mu : A \in \mathcal{A}\}$ is totally bounded.

■

We can now prove the Proposition of Orlicz-Pettis:

Proof. Let (x_n) be a sequence so that $\sum \epsilon_i x_i \mapsto x(\epsilon_i)$ for each $(\epsilon_i) \in \{-1, 1\}^{\mathbf{N}}$.

Define $f : \{-1, 1\}^{\mathbf{N}} \to E$ with $(\epsilon_i) \mapsto w - \sum_{i=1}^{\infty} \epsilon_i x_i$. f ist weakly continuous and bounded. (This follows from the Principle of uniform boundedness). $\{-1, 1\}^{\mathbf{N}}$ is a W-space. We can assume without loss of generality that $E = [\overline{x_n}]$ and so is separable. Hence f is Bochner integrable and so $f(\{-1, 1\}^{\mathbf{N}}) = \{\sum \epsilon_i x_i\}$ $|| \, ||$-compact.

■

Theorem 2 Krein-Milman theorem. *Let $B \subseteq E$ be weakly compact. Then $\overline{\Gamma}(B)$ is also weak compact.*

Proof.

1. Using the Eberlein smulian theorem, we can redue to the case where E is separable.

2. We claim that $\overline{\Gamma}(K) = \{\int \mathrm{Id} d\mu : \mu$ ein W-Maß on $K\}$ – ($\int \mathrm{Id}\mu$ is the barycentre of μ).

Since Id is $\sigma(E, E') - \sigma(E, E')$ continuous, it is weakly measurable and so $\|\ \|$-measurable (by the result of Pettis). Hence $\{\int_K \mathrm{Id} d\mu\}$ $\sigma(E, E')$ is compact.

∎

Proposition 8 *Let K be compact, $A \subseteq (C(K), \tau_p)$. If each sequence $\in A$ has a τ_p-cluster point, then A is τ_p-compact.*

Proof. We consider A as a subset of $(l^\infty(K), \tau_p)$. We can assume without loss of generality that $A \subseteq B_{l^\infty}$ (since A is weakly bounded and so $\|\ \|$-bounded). B_{ℓ^∞} is τ_p-compact (by Tychonov's theorem). Hence it suffices to show that $A \subseteq \overline{C(K)}$.

∎

3 Vector measures

Definition 5 *Let $(\Omega, \mathcal{A}_)$ be a set witj σ-algebra, E a Banach space. A* **finitely additive measure** *is a mapping $\mu : \mathcal{A} \to \mathcal{E}$, so that $\mu(\emptyset) = 0$, $\mu(A \cup B) = \mu(A) + \mu(B)$, ($A, B \in \mathcal{A}$ disjoint).*

IF

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

*(where (A_n) is a disjoint sequence of sets from $\mathcal{A}$), then μ is σ-***additive***.*

Examples.

1. Let $T : L^\infty[0,1] \to E$ continuous and linear. Then $\mu : A \mapsto T(\chi_A)$ is a finitely additive measure. μ is σ-additive, if T is γ-continuous.

2. Suppose that $T : L^1[0,1] \to E$ is continuous and linear. Then $\mu : A \mapsto T(\chi_A)$ is a σ-additive measure.

Let $\mu : \mathcal{A} \to \mathcal{E}$ be a measure. The **variation** of μ is

$$|\mu| := \sup \left\{ \sum_{i=1}^{n} ||\mu(A_i)|| : \Omega = \bigcup_{i=1}^{n} A_i \right\}$$

If $|\mu| < \infty$, then μ is of **bounded variation**.
Examples. Let $T : L^1 \to E$ be continuous and linear. Then $|\mu| < \infty$. For

$$\sum_{i=1}^{n} ||\mu(A_i)|| = \sum ||T(\chi_{A_i})|| \leq \sum ||T|| l(A_i) \leq ||T||.$$

The **semivariation** is

$$||\mu|| := \sup\{|f \circ \mu| : f \in B_{E'}\}$$

N.B. $||\mu|| \leq |\mu|$.
Examples. The measure

$$\begin{array}{rcl} \mu : \mathcal{A} & \to & L^\infty[0,1] \\ A & \mapsto & \chi_A \end{array}$$

has bounded semivariation but not bounded variation.

Lemma 1 *μ has bounded semivariation $\Leftrightarrow$ $\{\mu(A) | A \in \mathcal{A}\}$ is bounded in E.*

Proof. This is an application of the principal of uniform boundedness.

■

Examples. Each $T : L^\infty \to E$ induces a measure with bounded semivariation.
Examples. Let $\mathcal{A} = \{\mathcal{A} \subseteq \mathbf{N} : |\mathcal{A}| < \infty \vee |\mathbf{N} \setminus \mathcal{A}| < \infty\}$. This is a σ-algebra.
Put

$$\mu(A) = \{|A||A| < \infty|\mathbf{N} \setminus A||\mathbf{N} \setminus A| < \infty$$

Then μ is σ-additive but not of bounded variation.
We quote the following result on vector valued measures:

Proposition 9 *Let $\mu : \mathcal{A} \to \mathcal{E}$ be a measure of bounded variation. Then μ is σ-additive $\Leftrightarrow$ $|\mu|$ σ-additive.*

4 Integration

Let $\mu : \mathcal{A} \to \mathcal{E}$ be a measure, $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ a step function. Put $\int f d\mu := \sum \alpha_i \mu(A_i)$.

Lemma 2 *Let $\mu : \mathcal{A} \to \mathcal{E}$. Then*

$$||\mu|| = \sup \left\{ || \sum_{i=1}^n \epsilon_i \mu(A_i)|| : \mathcal{R} = \bigcup_{\rangle=\infty}^{\backslash} \mathcal{A}_\rangle \right\}$$

and further

$$\sup\{||\mu(A)|| : A \in \mathcal{A}\} \leq ||\mu||(\mathcal{R}) \leq \triangle \sup\{||\mu(\mathcal{A})|| : \mathcal{A} \in \mathcal{A}\}$$

If μ has bounded semivariation, then $T : f \mapsto \int f d\mu$ is continuous. For when f is a step function in $B_{\mathcal{L}^\infty}$ then

$$|| \int f d\mu|| \leq ||f||_\infty ||\mu||$$

and we can extend T to a bounded operator $T : \mathcal{L}^\infty(\mathcal{A}) \to \mathcal{E}$. Hence $L(\mathcal{L}^\infty(\mathcal{A}), \mathcal{E}) \cong \mathcal{M}^\lfloor(\mathcal{A}, \mathcal{E})$.

If ν is a W-Maß on $\mathcal{A}$, then

$$L(L^\infty(\nu), E) = M_\nu^b(\mathcal{A}, \mathcal{E}) = \{\mu \in \mathcal{M}^\lfloor : \nu(\mathcal{A}) = \prime \Rightarrow \mu(\mathcal{A}) = \prime\}$$

Definition 6 *μ is a ν-**absolutely continuous** vector measure*

$$\Leftrightarrow \bigwedge_{A \in \mathcal{A}} \nu(A) = 0 \Rightarrow \mu(A) = 0.$$

This is equivalent to the condition

$$\bigwedge_{\epsilon > 0} \bigvee_{¿0} \bigwedge_{A \in \mathcal{A}} \nu(A) \leq \delta \Rightarrow ||\mu(A)|| \leq .$$

Proposition 10 (Bartle-Dunford-Schwartz). *Let $\mu : \mathcal{A} \to \mathcal{E}$ be σ-additive (and so bounded). Then is $\{\mu(A) : A \in \mathcal{A}\}$ is relatively $\sigma(E, E')$-compact.*

Proof.

1. One uses the following result of James. Let E be a Banach space, $B \subseteq E$ bounded with the property
$$\bigwedge_{f \in E'} \bigvee_{x_0 \in E} f(x_0) = \sup\{f(x) : x \in B\}.$$
Then B is $\sigma(E, E')$-compact.

2. One uses properties of (L^∞, γ) and the fact that μ has a control measure i.e. there is a W-Maß ν, so that μ is absolutely continuous with respect to ν.

■

Proposition 11 *Let $T : L^\infty(\mu) \to F$ be linear. Then T is β-continuous $\Leftrightarrow f \circ T$ is continuous ($f \in F'$).*

This follows from the following property of the topology β.

Definition 7 *Let E, F be two vector spaces, which are in duality. Then the* **Mackey-topology** *$\tau(E, F)$ is the finest locally convexe topology on E, which is compatible with the duality.*

Proposition 12 *Let (E, τ) be a locally convex space so that $\tau = \tau(E, E')$. Then if $T : E \to F$ is a linear mapping with the property that $f \circ T$ is continuous for each $f \in F'$, then T is continuous.*

In order to show that a given Saks space is a Mackey space (i.e. that γ is the Mackey topology), one has to demonstrate that each $\sigma(E'_\gamma, E)$ weak compact set is γ-equicontinuous (i.e. that the converse of the roposition of Alaoglu gilt).

For this we use the following characterisation

Proposition 13 *Let $(E, ||\,||, \tau)$ be a Saks space. Then a bounded set $H \subseteq E'_\gamma$ is γ-equicontinuous $\Leftrightarrow \bigwedge_{\epsilon>0} \bigvee_{H_\epsilon \subseteq E'_\gamma}, H_\epsilon$ τ-equicontinuous and $H \subseteq H_\epsilon + \epsilon B_{||\,||}$.*

Examples. Let $E = C^b(S)$. $H \subseteq M_t(S)$ is β-equicontinuous $\Leftrightarrow H$ bounded and $\bigwedge_{\epsilon>0} \bigvee_{K \in \mathcal{K}(S)} \bigwedge_{\mu \in H} |\mu|(S \setminus K) \leq \epsilon$.

Proposition 14 *Let S be locally compact and paracompact. Then $(C^\infty(S), \beta)$ is a Mackey space.*

(the proof uses partitions of unity).

Proposition 15 *$(L^\infty(\mu), \underline{)}$ is a Mackey space.*

Proposition 16 *$(H^\infty(U), || \, ||, \tau_{\mathcal{K}})$ is not a Macky space.*
$(H^\infty(U), || \, ||, \tau_{L^1(\partial U)})$ is a Mackey space.

Proposition 17 *Let $\mu : \mathcal{A} \to \mathcal{E}$ be a σ-additive measure. Then there exists a W-measure $\nu : \mathcal{A} \to \mathbf{R}$ so that*

$$\bigwedge_{A \in \mathcal{A}} \nu(A) = 0 \Rightarrow \mu(A) = 0.$$

Corollar 2 *Let $\mu : \mathcal{A} \to \mathcal{E}$ be σ-additive. There exists a W-Maß ν, so that $T_\mu : L^\infty(\nu) \to E$ is continuous.*

Corollar 3 *$\mu : \mathcal{A} \to \mathcal{E}$ σ-additiv. Dann ist $\{\mu(A) : A \in \mathcal{A}\}$ relativ $\sigma(E, E')$ compact.*

Corollar 4 *Let $\mathcal{A}$ be a Boolean algebra, $\mu : \mathcal{A} \to \mathcal{E}$ finitely additive, so that for each $f \in E'$, $f \circ \mu$ is σ-additive. Then the following are equivalent*

1. *μ has a (norm-continuouose)σ-additive extension to a σ-algebra$\tilde{}A \subseteq \mathcal{A}$;*
2. *$\{\mu(A) : A \in \mathcal{A}\}$ is relatively weak-compact.*

Proposition 18 *Let $T : L^\infty(\nu) \to E$ be β-continuous. Then $\mu : \mathcal{A} \to \mathcal{E}$ is σ-additiv, where*

$$\mu : A \mapsto T(\chi_A).$$

On the other hand, let $\mu \ : \mathcal{A} \to \mathcal{E}$ be a vector-valued measure with control measure ν. Then integration induces a β-continuous operator $T : L^\infty(\nu) \to E$.

In order to show that T is β-continuous, we use the following property of (L^∞, β):

$$T : L^\infty \to E oxiscontinuous \Leftrightarrow f \circ T \in (L^\infty)' \quad (f \in E').$$

This follows from the Radon-Nikodym theorem and the fact that $(L^\infty, \beta)' = L^1$.

Proposition 19 *Let S be a $T_{31/2}$-space, $\mu : Bo(S) \to E$ $\sigma(E, E')$-Radon, i.e. $f \circ \mu$ is Radon for each $f \in E'$. Then μ is norm Radon.*

Proof of the extension theorem.

$1 \Rightarrow 2$ follows from the result of Bartle.

$2 \Rightarrow 1$: Let $C = \{\overline{\mu(A) : A \in \mathcal{A}}\}$. This set is weakly compact. Then $B = \overline{\Gamma(C)}$ is also weakly compact (Krein-Smulian). Consider the Saks space

$$(E_B, || \, ||_B, \sigma(E, E')).$$

Let ˜A = be the σ-algebra generated by $\mathcal{A}$. It has the representation

$$E = \overset{E}{\underset{\alpha \to \lim \ \alpha}{\longleftarrow}}$$

where the E_α are finite dimensional. Consider the diagram

$$\mathcal{A} \overset{\mu}{\to} \mathcal{E} \overset{\pi\beta}{\to} \mathcal{E}_\beta \overset{\pi_{\beta_\alpha}}{\to} \mathcal{E}_\alpha.$$

$\pi_{\beta \circ \mu}$ is σ-additive and so has an extension $\tilde{\mu}_\beta$. It follows from the uniqueness of the extension that $\pi_{\beta_\alpha} \circ \tilde{\mu}_\beta = \tilde{\mu}_\alpha$.

We define $\tilde{\mu}$ on ˜A by $\tilde{\mu}(A) := (\tilde{\mu}_\beta(A))_{\beta \in A}$. $\tilde{\mu}$ is $\sigma(E, E')$ σ-additive and so $|| \, ||$-additive (by the Theorem of Orlicz-Pettis).

■

Proposition 20 *Let $T : C(K) \to E$ be continuous and linear. Then there exists a representing measure μ :* Bo $(K) \to E''$*, where μ is $\sigma(E'', E')\sigma$-additive is (more precisely, a Radon measure with values in $(E'', || \, ||, \sigma(E'', E'))$. μ is $|| \, ||$-bounded and $\sigma(E'', E')$-compact regular. Then*

$$Tf = \int f d\mu \quad (f \in C(K)).$$

Proposition 21 *Let $T : C(K) \to E$ be continuous, with representing measure $\mu : Bo\,(K) \to E''$. then the following are equivalent*

1. T is weakly compact.

2. μ takes its values in E.

3. μ is σ-additive with respect to the norm.

Theorem 3 Vitali-Hahn-Saks theorem. *Let μ_n be a sequence of* **R**-*valued measures on $\mathcal{A}$, each of which is absolutely continuous with respect to a W-measure ν. Let $\mu_n(A)$ be convergent for each $A \in \mathcal{A}$. Then $\mu_0 : A \mapsto \lim_{n\to\infty} \mu_n(A)$ is absolutely continuous with respect to ν. Let μ_n be a sequence of σ-additive measures on $\mathcal{A}$ so that $\mu_n(A)$ converges for each $A \in \mathcal{A}$. Then $\mu_0 : A \mapsto \lim_{n\to\infty} \mu_n(A)$ is σ-additive.*

We now sketch briefly how one can use the methods of Saks space theory to obtain a series of connected results. We begin with a famous Proposition of Rainwater:

Proposition 22 (Rainwater.) *Let (x_n) be a sequence in a Banach space E. Then $x_n \stackrel{\sigma(E,E')}{\rightarrow} x$ if and only if (x_n) is bounded and $f(x_n) \rightarrow f(x) \quad (f \in Ex(B_{E'})$.*

This is proved using the

Theorem 4 Theorem of Choquet. *Let E be a locally convex space, $B \subseteq E$ metrisable, compact and convex. Then $Ex(B)$ is a G_δ-set and each $x \in B$ is the barycentre of a W-measures μ on $Ex(B)$ i.e. $x = \int Id\, d\mu$.*

PROOF. We show how to deduce the result of Rainwater from Choquet's theorem. We can suppose that E is separable. Then $B = B_{E'}, \sigma(E', E))$ satisfies the hypotheses. Hence we can embed E in $C^\infty(\mathrm{Ex}\,(B_{E'}))$. This is an isometry (by the Krein-Milman theorem). For each $x \in E$ is affine on $B_{E'}$ and so assumes its supremum on an extreme point.

By Choquet's theorem

$$\begin{aligned}(E, \sigma(E, E')) &\subseteq (C^\infty(\mathrm{Ex}\,(B_{E'}), \sigma(C^\infty, C^{\infty\prime}_\beta))\\ (E, \sigma(E, Ex(B_{E'}))) &\subseteq C^\infty(\mathrm{Ex}\,(B_{E'}), \tau_p),\end{aligned}$$

where each inclusion is an isomorphism.

■

We now use a version of the following characterisation of weak convergence in $C(K)$-spaces:

Proposition 23 (Grothendieck.) *Let x_n be a sequence in $C(K)$. Then*

$$x_n \stackrel{\sigma(C(K),C(K)'}{\rightarrow} x \Leftrightarrow x_n \textit{ bounded and } x_n(t) \rightarrow x(t) \quad (t \in K).$$

It is trivial to derive the following version for $C^\infty(S)$-spaces:

Proposition 24 *Let x_n be a sequence in $C^\infty(S)$. Then*

$$x_n \stackrel{\sigma(C^\infty(S),C^\infty(S)''}{\rightarrow} xx \Leftrightarrow x_n oxboundedandx_n(t) \rightarrow x(t) \quad (t \in S)$$

Proof of Rainwater's We consider x_n als a sequence in $C^\infty(\mathrm{Ex}\,(B_{E'}))$ and and use the generalised version of Grothendieck's result.

■

In a similar manner one can prove the following Proposition:

Proposition 25 (Fremlin, Bourgain, Talagrand) *Let $B \subseteq E$ be bounded. Then B is $\sigma(E, E')$-compact if and only if B is $\sigma(E, Ex B_{E'})$-compact resp. each sequence in B has a $\sigma(E, Ex B_{E'})$ cluster point.*

PROOF. Step 1: Using the classical Eberlein-Smulian theorem we can reduce to the case where E is separable.

Step 2: If S is a suitable space (i.d. so that $(C^\infty(S), \beta)$ is complete and metrisable). Then a bounded subset $B \subseteq C^\infty(S)$ is weakly compact $\Leftrightarrow$ each sequence $x_n \in B$ has a τ_p-cluster point.

Step 3: is as in the proof of Rainwater's result.

■

4.1 Topologies on operator spaces

We now describe some natural topologies on operator spaces and discuss some applications, in particular, a proof of the spectral theorem for unbounded self adjoint operators, which uses the gneralised Riesz representation theorem.
PROOF. **Proof of the spectral theorem for bounded operators.** Let T be self-adjoint. We construct the functional calculus $\Phi : p \mapsto p(T)$ for polynomials and show that $\Phi : \mathrm{Pol}([\alpha, \beta]) \subseteq C[\alpha, \beta] \to L(H)$ is continuous. Φ can be extended to a continuous linear mapping $C[\alpha, \beta] \to L(H) = (L(H), || \, ||, \tau_s)$. By the Riesz representation theorem there exists $\mu : \mathrm{Bo}([\alpha, \beta]) \to L(H)$ with the property that $\Phi(f) = \int_\alpha^\beta f(x) d\mu(x)$. It is then easy to obtain the classical formulation of the spectral theorem.

■

EXAMPLES. Show that $\mu(A)$ is an orthogonal projection for $A \in \mathrm{Bo}([\alpha, \beta])$.

We mention briefly that one can extend this proof to the unbounded case using Saks space method. Let $T : H \to H$ be a p.l.o. We construct a functional calculus $C^\infty(\mathbf{R}) \to L(H)$ by means of the following Lemma:

Lemma 3 *Let $T : H \to H$ be a s.a.p.l.o. Then there exists a sequence (H_n) of closed subspaces so that*

1. *$D(T) \cap H_n = H_n$ and $T(H_n) \subseteq H_n$;*
2. *$H_n \subseteq H_{n+1}$ and $\bigcup_{n=1}^\infty H_n$ is dense in H.*

The proof of the spectral theorem is then as above, whereby one uses the following topologies on $L(H)$.

τ_w — the locally convex topology generated by the seminorms $\{p_{x,y} : T \mapsto |(Tx|y)| : x, y \in H\}$;

τ_s — generated by $\{p_x : T \mapsto ||Tx||(x \in H)\}$;

τ_s^* — generated by $\{p_x\}$ and $\{p_x^*\}$, where

$$p_x^* : T \mapsto ||T^*x||.$$

These topologies are not complete. Hence we replace them by the corresponding mixed topologies:

$$\beta_\sigma = \gamma(|| \, ||, \tau_w), \beta_s = \gamma(|| \, ||, \tau_s), \beta_{s^*} = \gamma(|| \, ||, \tau_s^*).$$

They are all complete. (N.B. sequential convergence is as for $\tau_w, \tau_s, \tau_{s^*}$—this follows form the principle of uniform boundedness.

When H is separable, then

$$H = \overline{\bigcup_{n \in \mathbf{N}} H_n},$$

where $\dim H_n = n$ and so

$$\begin{aligned} (L(H), || \, ||, \tau_w) &= S - \lim L(H_n) \\ (L(H), || \, ||, \tau_s) &= S - \lim L(H_n, H). \end{aligned}$$

Definition 8 *The space $N(H)$ of nuclear operators is defined as follows:*

$$N(H) = \{T \in K(H) : \sum_{n=1} l_n(VT^*T) < \infty\}.$$

This coincides with the set of all $T \in L(H)$, so that T has a faktorisation $T_1.T_2$ with T_1 and T_2 Hilbert-Schmidt operators.

Then

$$(L(H), \beta_\sigma \beta_s \beta_{s'})' = N(H)$$

Further $\beta_\sigma = \sigma(L(H), N(H))$ on $B_{L(H)}$.

Proposition 26 (Akemann) *β_{s^*} is the Mackey-topology on $L(H)$.*

5 Saks spaces

The object of study in this chapter are Banach spaces with a supplementary structure in the form of an additional locally convex topology. The motivation lies in the interplay between certain mathematical objects (topological spaces,measure spaces etc.) and suitable spaces of (complex-valued) functions on them. These often have a natural Banach space structure. However, by passing over from the original spaces to the associated Banach spaces, one frequently loses crucial information on the underlying space. A good example (which will be the subject of our most important application of mixed topologies) is the Banach space $C^\infty(S)$ of bounded, continuous, complex-valued functions on a locally compact space S where it is impossible to recover S from the Banach space structure of $C^\infty(S)$ (in contrast to the case of compact spaces S).

As we shall see, this situation can be saved by enriching the structure of $C^\infty(S)$ with the topology $\tau_\mathcal{K}$ of uniform convergence on the compact subsets of S. The class of spaces that we consider can be regarded as a generalisation of the class of Banach spaces (we can "enrich" a Banach space in a trivial way, namely by adding its own topology). In fact these spaces can be regarded as projective limits of certain spectra of Banach spaces with contractive linking mappings (just as one can regard (complete) locally convex spaces as projective limits of arbitrary spectra of Banach spaces) and we shall lay particular emphasis on this fact for two reasons: for purely technical ones and secondly because, in applications to function spaces, we shall constantly use the fact that our function spaces are constructed out of simpler blocks which correspond exactly to the members of a representing spectrum of Banach spaces. As an example, dual to the fact that one can consider a locally compact space as being built up from its compact subspaces, we find that one can construct the space $C^\infty(S)$ from the spectrum defined by the spaces $\{C(K)\}$ as K runs through these subsets.

One of our main tools in the study of our enriched Banach spaces will be a natural localy convex topology – the mixed topology of the title of this chapter.

For the convenience of the reader, we now give a brief summary of this chapter. In the first section, we give a basic treatment of generalised inductive limits. Essentially, we consider a vector space with two locally convex topologies which satisfy suitable compactibility conditions. We then introduce in a natural way a "mixed topology" and this section is devoted to relating its properties to those of the original topologies. However, a closer examination of the definitions and results shows that, for one of the topologies, only the bounded sets are relevant. We have taken the consequences of this observation by replacing the topology by a "bornology", that is, a suitable collection of sets which satisfy the properties which one would expect of a family of bounded sets. We really only use the language of bornologies and introduce explicitly all of the terms which we use. In section 2, we give a list of examples spaces with mixed topologies. Some of these will be studied detail (and more generality) in the following chapters. Other are introduced to supply counter-examples. All are used to illustrate the ideas of the first section. In section three, we define the class of enriched Banach spaces mentioned, restate the results of section 1 in the form that we shall require them for applications and describe the usual methods for constructing new spaces (subspaces, products, tensor products etc.). It is perhaps not inappropriate to mention here that one of the main reasons for our emphasis on spaces with two structures (a norm **and** a locally convex topology) rather that on locally convex spaces of a rather curious type is the fact that it is important that these constructions be carried out such a way that this double structure is preserved.

6 Basic theory

As announced in the Introduction to this chapter, it is convenient for us to use the language of bornologies.

Definition 9 *Let E be a vector space. A* **Ball** *in E is an absolutely convex subset of E which does not contain a nontrivial subspace. If B is a ball in E, we write E_B for the linear span $\bigcup_{n=1}^{\infty}$ if B in E. Then*

$$|| \, ||_B : x \to \inf\{l > 0 : x \in lB\}$$

is a norm on E. If $(E_B, || \, ||_B)$ is a Banach space, B is a **Banach ball***.*

Note that any absolutely convex, bounded subset of a locally convex space is a ball. The following Lemma gives a sufficient (but not necessary) condition for it to be a Banach ball.

Lemma 4 *Let B be a bounded ball in a locally convex space (E, τ). Then if B is sequentially complete for τ (and in particular if its is τ-complete), B is a Banach ball.*

PROOF. Let (x_n) be a Cauchy sequence in $(E_B, || \, ||_B)$. Then, since B is bounded, (x_n) is τ-Cauchy. Hence there is an $x \in B$ so that $x_n \to x$ for τ. We show that $||x_n - x||_B \to 0$. If $\epsilon > 0$, there is an $N \in \mathbf{N}$ so that $(x_m - x_n)$ belongs to ϵB for $m, n \gamma N$. Since B (and so also ϵB) is sequentioally complete and so sequentioally closed, we can take the limit over n to deduce that $x_m - x$ belongs to ϵB for $m \geq N$.

■

Recall that if E is a vector spaces with a (**convex**) **bornology** $\mathcal{B}$, then a subset B of E is **$\mathcal{B}$-bounded** if it is contained in some ball in $\mathcal{B}$.

A **basis** for $\mathcal{B}$ is a subfamily $\mathcal{B}_\infty$ of $\mathcal{B}$ so that each n$B \in ?$ is a subset of some $B_1 \in \mathcal{B}_\infty$.

$(E, \mathcal{B})$ is **complete** if $\mathcal{B}$ has a basis consisting of Banach balls.

$\mathcal{B}$ is of **countable type** if $\mathcal{B}$ has a countable basis. If (E is a locally convex space, then $\mathcal{B}_\tau$, the family of all τ-bounded, absolutely convex subsets of E, is a bornology on E – the **von Neumann bornology**. In many of our applications $\mathcal{B}$ will be the von Neumann bornology of a normed space $(E, || \, ||)$. This is of countable type (the family $\{nB_{|| \, ||}\}_{n \in \mathbf{N}}$ where $B_{|| \, ||}$ is the unit ball of E is a basis).

We now consider a vector space E with a locally convex topology τ and a bornology $\mathcal{B}$ of countable type which are **compatible** in the following sense:

$\mathcal{B} \subseteq \mathcal{B}_\tau$ and $\mathcal{B}$ has a basis of τ-closed sets. Then we can choose a basis (B_n) for $\mathcal{B}$ with the following properties:

(a) $B_n + B_n \subseteq B_{n+1}$ for each n;

(b) each B_n is τ-closed.

(If (C_n) is a countable baisis for $\mathcal{B}$, we can define (B_n) inductively as follows: take $B_1 = C_1$. Once $B_1, \dots, B_n$ have been chosed, we can find a τ-closed ball in $\mathcal{B}$ which contains $B_n + B_n + C_{n=1}$. This is our $B_{n=1}$). Infuture, we shall tacitly assume that a given basis (B_n) has the above properties.

Definition 10 *We define the mixed locally convex structure $\gamma = \gamma[\mathcal{B}, \tau]$ as follows:*

Let $\mathcal{U} = (\mathcal{U}_\backslash)_{\backslash=\infty}^{\infty}$ be a sequence of absolutely convex τ-neighbourhoods of zero and write

$$\gamma(\mathcal{U}) := \bigcup_{\backslash=\infty}^{\infty} (\mathcal{U}_\infty \cap \mathcal{B}_\infty + \cdots + \mathcal{U}_\backslash \cap \mathcal{B}_\backslash).$$

Then the set of all such $\gamma(\mathcal{U})$ forms a base of neighbourhoods of zero for a locally convex structure on E and we denote it by $\gamma[\mathcal{B}, \tau]$ (or simply by γ if *no confusion is possible). In the case where $\mathcal{B}$ is the bornology defined by a norm on E, we write $\gamma[|| \, ||, \tau]$ for the structure $\gamma[\mathcal{B}, \tau]$.*

The following Proposition gives a natural characterisation of γ.

Proposition 27 *(i) γ is finer that τ;*

(ii) γ and τ coincide on the sets of $\mathcal{B}$;

(iii) γ is the finest linear topology on E which coincides with τ on the sets of $\mathcal{B}$.

PROOF.

(i) if U is a τ-neighbourhood of zero, then $U \subseteq \gamma(U_n)$ where $U_n := 2^{-n}U$.

(ii) if $B \in \mathcal{B}$, we can choose a positive integer r so that $B - B \subseteq B_r$. A typical neighbourhood of the point $x_0 \in B$ for the topology induced by γ on B has the form

$$B \cap (x_0 + \gamma(U_n)).$$

Then $U_r \cap (B - B) \subseteq U_r \cap B_r \subseteq \gamma((U_n))$ and so

$$(x_0 + U_r) \cap B \subseteq (x_0 + \gamma((U_n)) \cap B.$$

(iii) let τ_1 be a linear topology on E which coincides τ on the sets of $\mathcal{B}$. We show that γ is finer than τ_1. W be a neighbourhood of zero for τ_1 and choose neighbourhood W_n of zero so that $W_0 = W$ and $W_n + W_n \subseteq W_{n-1}$ $(n \geq 1)$. There are τ-neighbourhoods (U_n) of zero so that $U_n \cap B_n \subseteq W_n$. Then, for any n

$$(U_1 \cap B_1) + \cdots + (U_n \cap B_n) \subseteq W$$

and so $y((U_n)) \subseteq W$.

■

Corollar 5 *(i) γ is independent of the choice of basis (B_n);*

(ii) if τ and τ_1 are suitable locally convex topologies E (i.e. if τ and τ_1 are compatible with $\mathcal{B}$) then $\gamma[\mathcal{B}, \tau] = [\gamma[\mathcal{B}, \tau_\infty]$ if and only if τ and τ_1 coincide on the sets of $\mathcal{B}$.

The locallisation property of γ expressed in 27 implies that the continuity of linear mappings is determined by their behaviour on the bounded sets of E.

Corollar 6 *Let H be a family of linear mappings from E into a topological vector space F. Then H is γ-equicontinuous if and only if $H|_B$ is τ-equicontinuous for each $B \in \mathcal{B}$. In particular, a linear mapping T from E into F is continuous if and only if $T|_B$ is τ-continuous for each $B \in \mathcal{B}$.*

PROOF. We must show that if W is a neighbourhood of zero in F, x a point of B. We must find a neighbourhood U of zero in E so that $T((x+u) \cap B) \subseteq Tx + V$. We choose U (absolutely convex) so that $T(B \cap (U/2)) \subseteq V/2$. Then if $y \in ((x+U) \cap B)$, $x - y \in B - B = 2B$ and the result follows from the inclusion $T(2B \cap U) \subseteq V$.

■

As we shall see later, the topology γ is, in the interesting cases, never metrisable (or even bornological). However, it does, sometimes, have one useful property in common with such spaces.

Proposition 28 *Suppose that $\mathcal{B}$ has a basis of τ-metriesable sets. Then a linear mapping from E into a topological vector space F is continuous if and only if it is sequentially continuous.*

PROOF. For any $B \in \mathcal{B}$, $T|_B$ is sequentially continuous and continuous. The result then follows from 6.

■

In the following Proposition, we characterise certain properties (boundedness, compactness, convergence) with respect t γ directly in terms of $\mathcal{B}$ and τ.

■

Proposition 29 *A sequence (x_n) in E converges to x (E, γ) if and only if $\{x_n\}$ is $\mathcal{B}$-bounded and $x_n \to x$ in (E, τ).*

PROOF. We can suppose that $x = 0$. By 27 it suffices to that if $x_n \to 0$ in (E, τ), then $\{x_n\}$ is $\mathcal{B}$-bounded. If t were false, we could find a subsequence (x_{n_k}) so that $x_{n_k} \notin B_k$. Since B_k is τ-closed, we can choose a τ-neighbourhood of zero U_k so that $x_{n_k} \notin B_k 2U_k$ and we can suppose that $U_k + U_k \subseteq U_{k-1}$ $(k > 1)$. Then for each $k > 1$

$$\begin{aligned} \gamma((U_n)) &= \bigcup_{n=1}^{\infty} (U_1 \cap B_1 + \cdots + U_n \cap B_n) \\ &\subseteq \bigcup_{p=1}^{\infty} (B_1 + \cdots + B_{k-1} + U_k + \cdots + U_{k+p}) \\ &\subseteq B_k + 2U_k. \end{aligned}$$

■

Hence $x_{n_k} \notin \gamma((U_n))$ for each k, which contradicts the fact that $x_n)$ is a g-null-sequence.

Proposition 30 *A subset B of E is γ-bounded if and only if it is $\mathcal{B}$-bounded.*

PROOF. Suppose that B is $\mathcal{B}$-bounded. Then $B \subseteq B_r$ for some r. Let (U_n) be a sequence of absolutely convex neighbourhoods of zero. Then there is a $K > 1$ so that $B \subseteq KU_r$. Hence

$$B \subseteq K(U_r \cap B_r) \subseteq K\gamma((U_n))$$

and so B is γ-bounded.

■

Now suppose that B is γ-bounded. If B were not $\mathcal{B}$-bounded, we could find a sequence (x_n) in B so that $x_n \notin nB_n$ for each positive integer n. Now $n^{-1}x_n \to 0$ in (E, γ) and so $\{n^{-1}x_n\}$ is $\mathcal{B}$-bounded – contradiction.

Proposition 31 *A subset A of E is γ-compact (precompact, relatively compact) if and only if it is $\mathcal{B}$-bounded and τ-compact (precompact, relatively compact).*

PROOF. This follows immediately from 30 and 27 (ii).

■

We recall that a locally convex space is **semi-Montel** if its bounded sets are relatively compact. It is **Montel** if, in addition, it is barreled. In the next Proposition, we characterise semi-Montel mixed spaces. As we shall se below, non-trivial mixed topologies are never barrelled – and so never Montel.

Proposition 32 $(E_,\gamma)$ *is semi-Montel if and only if $\mathcal{B}$ has a basis of τ-compact sets.*

PROOF. This is a direct consequence of 1.12 and the definition.

■

We now consider the completeness fo (E, γ). It follows from completeness theorem of RAIKOV which generalises KÖTHE's completeness theorem that (E, γ) is complete if and only if $\mathcal{B}$ has a basis of τ-complete sets. However, this result is rather inaccessible and we shall give a proof based on duality theorem later.

In the case where $\mathcal{B}$ is the bornology associated with a norm on E there are three natural locally convex topologies τ, and $\tau_{\|\ \|}$, the norm topology, on E and we discuss their distinctness. We first note that the equality $\tau = \gamma$ means essentially that τ is already a mixed topology i.e. we have gained nothing by mixing. On the other hand, the condition $\gamma = \tau_{|}| \, ||$ means that we are in the trivial situation (trivial “enriched” by its own topology (since $\gamma[|| \, ||, \tau] = \gamma[|| \, ||, \tau_{|}| \, ||]$. The following result shows that if γ belongs to the traditional classes of well-behaved locally convex spaces, then we have this trivial situation.

Proposition 33 *If γ is bornological (in particular, metrisable) or barrelled, then $\gamma = \tau_{|}| \, ||$.*

PROOF. If γ is bornological, then the identity mapping from (E, γ) into $(E, || \, ||)$, being bounded 30, is continuous and so $\gamma \subseteq \tau_{|}| \, ||$. The converse inequality is obvious. If γ is barrelled, then $B_{|}| \, ||$, the unit ball of $(E, || \, ||)$, being a barrel in (E, γ), is a γ-neighbourhood of zero (we are assuming that $B_{|}| \, ||$ is τ-closed – strictly speaking, this need not to be the case. However, it follows easily from the compatibility conditions that we can find an equivalent norm so that this condition is satisfied).

■

The essential property of γ is given in 27 (iii). The neighbourhood basis used in the definition was chosen so that this would hold. However, in applicaitons, we shall frequently require a much less obvious description of γ-neighbourhoods of zero. Let $\mathcal{U}$ be as in 10 (except that it is now convenient to index from zero to infinity) and put

$$\tilde{\gamma}(\mathcal{U}) := \mathcal{U}_{\prime} \cap \bigcup_{\backslash=\infty}^{\infty} (\mathcal{U}_{\backslash} + \mathcal{B}_{\backslash}).$$

Then the family of such sets forms a neighbourhood basis for a locally convex topology on E which we denote by $\tilde{\gamma}[\mathcal{B}, \tau]$.

Proposition 34 $\tilde{\gamma}[\mathcal{B}, \tau] = \gamma[\mathcal{B}, \tau]$.

PROOF. Firstly, $\tilde{\gamma}$ is coarser than τ on each set B_k. For $\tilde{\gamma}(\mathcal{U}) \cap \mathcal{B}_{||} = (\mathcal{U}_{\prime} \cap \bigcap_{\backslash=\infty}^{||} (\mathcal{B}_{\backslash} + \mathcal{U}_{\backslash})) \cap \mathcal{B}_{||}$ and this is a $\tau|_{B_k}$-neighbourhood of zero. Hence by 27 (iii), γ is finer that $\tilde{\gamma}$. Now we show that γ is coarser than $\tilde{\gamma}$. Let $\gamma((U_n))$ be a typical γ-neighbourhood of zero. There exists a decreasing sequence $(V_n)_{n=0}^{\infty}$ of τ-neighbourhood of zero so that $\tilde{\gamma}((V_n)) \subseteq \gamma((U_n))$. Choose $x \in \tilde{\gamma}((V_n))$. Then $x \in V_0$ and, for each n, x has a decomposition $y_n + z_n$ where $y_n \in B_n$, $z_n \in V_n$. Define $z_1 := y_1$, $x_n := y_n - y_{n-1}$ $(n > 1)$. Then

$$x_1 + \cdots + x_n + z_n = y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) + z_n = y_n + z_n = x$$

and $z_{n-1} = x_n + z_n$.

■

We have $x_n = z_{n-1} - z_n \in V_{n-1} + V_n \subseteq V_n + V_n \subseteq U_{n+1}$ and

$$x_n = y_n - y_{n-1} \in B_n + B_{n+1} \subseteq B_{n+1}.$$

Hence $x_n \in U_{n+1} \cap B_{n+1}$.

If n_0 is so chosen that $x \in B_{n_0}$, then $z_{n_0} = x - y_{n_0} \in B_{n_0}$. On the other hand, $z_{n_0} \in V_{n_0} \subseteq U_{n_0} + 2$. Hence we have

$$\begin{aligned} x = x_1 + \cdots + x_n + z_n &\in U_2 \cap B_2 + \cdots + U_{n_0+1} \cap B_{n_0+1} + U_{n_0+2} \cap B_n \\ &\subseteq \gamma((U_n)). \end{aligned}$$

Corollar 7 *γ has a basis consisting of τ-closed sets.*

PROOF. If U_n is an absolutely convex τ-neighbourhood of zero then

$$(2^{-1}U_n) + B_n \subseteq \overline{(2^{-1}U_n) + B_n} \subseteq U_n + B_n$$

and so $\tilde{\gamma}((2^{-1}U_n)) \subseteq U_0 \cap \bigcap_{n=1}^{\infty} \overline{((2^{-1}U_n) + B_n)} \subseteq \tilde{\gamma}((U_n))$ and this implies the result.

We now consider duality for (E, γ). E has three dual space

E'_τ – the dual of the locally convex space (E, τ);

E'_γ – the dual of the locally convex space (E, γ);

$E'_\mathcal{B}$ – the dual of the bornological space $(E, \mathcal{B})$, that is the space of linear forms on E which are bounded the sets of $\mathcal{B}$.

Then $E'_\tau \subseteq E'_\gamma \subseteq E'_\mathcal{B}$ and we regard each of these spaces as a cally convex space with the topology of uniform vonvergence the τ-bounded sets (resp. the γ-bounded sets, resp. the sets $\mathcal{B}$. Since $\mathcal{B}$ is of countable type, $E'_\mathcal{B}$ is metrisable and is also clearly complete (since the uniform limit of bounded functions is bounded). Hence it is a Fréchet space. Our next result characterise E'_γ and its equicontinuous subsets in terms of E'_τ and $E'_\mathcal{B}$. Note that this result is a special case of Grothendieck's completeness theorem.

■

Proposition 35 • *(i) E'_γ is a locally convex subspace of $E'_\mathcal{B}$;*

(ii) E'_γ is the closure of E'_τ in $E'_\mathcal{B}$ and so is a Fréchet space.

Proof.

(i) follows directly from 30.

(ii) E'_γ is closed in $E'_\mathcal{B}$ since the limit of a sequnce in $'_\gamma$ is continuous on the sets of $\mathcal{B}$ and so is in E'_γ by 6. we show that $E'\tau$ is dense in E'_γ. Let B be a τ-closed ball in $\mathcal{B}$ and ϵ be a positive number. If $f \in E'$ then there is an absolutely convex τ-neighbourhood U of zero so that $|f(x)| \leq \epsilon$ if $x \in B \cap U$ i.e. f belongs to $\epsilon(B \cap U)^0$ (polar in E^*, the algebraic dual of E). Now the polar of $B \cap U$ is the closure of $1/2(B^0 + U^0)$ in $\sigma(E^*, E)$. But this set is closed since U^0 is $\sigma(E^*, E)$-compact by the theorem of ALAOGLU-BOURBAKI and so

$$(B \cap U)^0 \subseteq B^0 + U^0.$$

Hence f belongs to $\epsilon(U^0 + B^0)$ and so there is a g belonging to $\epsilon U^0 \subseteq E'_\tau$ such that $f - g$ belongs to ϵB^0 i.e. $|f(x) - g(x)| < \epsilon$ if $x \in B$.

■

Corollar 8 *Let τ_1, τ_2 be locally convex topologies on E which are compatible with $\mathcal{B}$ and suppose that τ_1 and τ_2 have the same dual. Then $\gamma[\mathcal{B}, \tau_\infty]$ and $\gamma[\mathcal{B}, \tau_\in]$ have the same dual.*

Proposition 36 *A subset B of E is γ-weakly compact if and only if it is $\mathcal{B}$-bounded and $\sigma[E, E'_\tau]$-compact.*

Proof. The condition is clearly necessary. It is sufficient since if B is $\mathcal{B}$-bounded then, regarded as a subset of the dual of E'_γ, it is equicontinuous and so the weak topologies defined by $E'\gamma$ and its dense subspaces E'_τ coincide on it.

■

Corollar 9 *(E, γ) is semi-reflexive if and only if $\mathcal{B}$ has a basis of $\sigma(E, E'_\tau)$-compact sets.*

Proposition 37 *A subset H of $E'_\mathcal{B}$ is γ-equicontinuous if and only if it satisfies the following condition:*

> *For every strong neighbourhood U of zero in E'_γ, there is a τ-equicontinuous set H_1 in E'_τ so that*
>
> $$H \subseteq U + H_1.$$

PROOF. Sufficiency: choose $B \in \mathcal{B}$, $\epsilon > 0$. It is sufficient t find a τ-neighbourhood V of zero so that if $x \in B \cap V$, $f \in ?$ then $|f(x)| \leq \epsilon$. We choose V so that

$$H \subseteq (\epsilon/2)B^0 + (\epsilon/2)V^0 \subseteq \epsilon(V \cap B)^0$$

(for the last inclusion, cf. the proof of 35 (ii)).

Necessity: suppose that H is γ-equicontinuous and U is a strong neighbourhood of zero in E'_γ. We can suppose that $U = B_k^0$ for some positive integer k. Then there is γ-neighbourhood of zero $\gamma((U_n))$ so that

$$\begin{aligned} H \quad \subseteq \quad & \{\gamma((U_n))\}^0 \quad (U_1 \cap B_2 + \cdots + U_n \cap B_k)^0 \\ & (U_k \cap B_k)^0 \quad U_k^0 + B_k^0. \end{aligned}$$

■

Corollar 10 *Let (x_n) be a null-sequence in E'_γ. Then $\{x_n\}$ is γ-equicontinuous.*

PROOF. If U is a strong neighbourhood of zero in $E'_\mathcal{B}$, then all but finitely many of the elements of the sequence lie in U. Hence we can apply 37.

■

Corollar 11 *Let A be a compact subset of E'_γ. Then A is γ-equicontinuous.*

PROOF. Since E'_γ is a Fréchet space, A is contained in the closed, absolutely convex hull of a null sequence. By 10 the range of this sequence is equicontinuous and hence so is its $\sigma(E'_\gamma, E)$-closed, absolutely convex hull and this set contains A.

■

If (F, τ_1) is a locally convex space and E is a subspace, there is a natural vector space isomorphism between F'/E^0 and E' induced by the restriction mapping from f' onto E'. Since the latter mapping is continuous for the strong topologies, the map from F'/E^0 onto E' is continuous. In general, it is not a locally convex isomorphism. This is the case, however, when E has a mixed topology.

Proposition 38 *Let (E, γ) be a locally convex subspace (F, τ_1). Then the strong topology on F'/E^0 as the dual of (E, γ) coincides with the quotient of the stron topology on?*

Proof. We show that the natural mapping from E'_γ onto F'/E^0 is continuous. Since E'_γ is a Fréchet space, it suffices to show that every sequence which converges to zero in E'_γ is bounded in F'/E^0. But such a sequence of an equicontinuous set in (Hahn-Banach theorem). The image of such a set in F'/E^0 is bounded.

■

Corollar 12 *If, in addition, E is dense in F, then every bounded set in F is containded in the closure of a bounded set in E.*

Proof. By 38, the natural vector space isomorphism between the duals of E and F is an isomorphism for the strong topologies and this is equivalent to the statement of the Corollary (bipolar theorem).

■

As an immediate Corollary, we have

Proposition 39 *The space (E, γ) is complete if and only if $\mathcal{B}$ has a basis of τ-compact sets.*

6.1 Examples

A. Let X be a completely regular space, S a collection of sets of X so that $\cup\mathcal{S}$ is dense in X. We denote by $C^\infty(?)$ the space of bounded, continuous functions from X into $\mathbf{C}$. Then

$$|| \, ||_\infty : X \to \sup\{|x(t)| : t \in X\}$$

is a norm on $C^\infty(X)$ and $(C^\infty(X), || \, ||_\infty)$ is a Banach space. If $A \subseteq X$ then

$$p_A : x \to \sup\{|x(t)| : t \in X\}$$

is a seminorm on $C^\infty(X)$ and we denote by $\tau_{\mathcal{S}}$ the locally convex structure generated by $\{p_A : A \in \mathcal{S}\}$.

Then $(C^\infty(X), || \, ||_\infty, \tau_{\mathcal{S}})$ satisfies the conditions of ?? [1.3 ist durgestrichen] and so we can form the mixed topology $\gamma[|| \, ||_\infty, \tau_{\mathcal{S}}]$ which we denote by $\beta_{\mathcal{S}}$.

B. Let G be an open subset of the complex plane and denote by $H^\infty(G)$ the subspace of $C^\infty(G)$ consisting of holomorphic functions. Then $(H^\infty(G), || \, ||_\infty, \tau_{\mathcal{K}})$ (where $\mathcal{K}$ is the family of compact subsets of G and $|| \, ||_\infty$ and $\tau_{\mathcal{K}}$ are defined as in D)?? satisfies the conditions of 1.3?? Wedenote the corresponding mixed topology by β.

C. Let F be a Fréchet space, G a Banach space and denote by E the space $L(F, G)$ of continuous linear mappings from F into G. If $\mathcal{B}$ is a bornology, we define on E the following structures:

$\mathcal{B}_{\rceil\amalg\sqcap}$ – the bornology generated by the equicontinuous balls in E;

$\tau_{\mathcal{B}}$ – the topology of uniform convergence on the sets of $\mathcal{B}$.

Then the conditions of 1.3 ?? are satisfied.

D. Let (Ω, Σ, μ) be a measure space i.e. Σ is a σ-algebra on the set Ω and μ is a positive, σ-finite, n σ-additive measure on Σ. $L^\infty(\mu)$ denotes the space of equivalence classes of μ-essentially bounded measurable functions on Ω. We regard it as a Saks space with

a) the supremum norm $|| \, ||_\infty$;

b) the topology τ_1 defined by the seminorm

$$p_A^1 : x \to \int_A |x| d\mu$$

where A runs through the family Σ_0 of subsets of Σ of finite measure. (Sometimes it is convenient to consider a variant of this situation where X is a locally compact space, Σ is the Borem algebra of X and μ is a Radon measure. In this case the condition that μ be σ-finite can be droppend).

E. Let F and G be Banach spaces and E be the space $L(E, F)$ of bounded linear operators from E in F. On this space we consider the operator norm $|| \, ||$ and the weak resp. strong operator topologies τ_σ resp. τ_s defined by the seminorms $\{p_{f,x} : f \in G', x \in E\}$ resp. $\{p_x : x \in E\}$ where

$$p_{f,x} : T \to |f(Tx)| \quad (f \in F', x \in E)$$

resp. $p_x : T \to ||Tx||$.

If $F = G = H$ (a Hilbert space) we also consider the strong* topology τ_{s^*} defined by the seminorms $\{p_x : x \in H\} \cup \{p^* : x \in H\}$ where

$$p_x^* : T \to ||T^* x||.$$

F. Let H be a Hilbert space, $\{H_\alpha\}_{\alpha \in A}$ a family of subspaces which is directed on the right (i.e. is such that if H_α and H_β are members of the family, then there is a $\gamma \in A$ so that H_γ contains H_α and H_β) and has dense union. We consider the subspace E of $L(H)$ consisting of those operators $T \epsilon L(H)$ which are reduced by $\{H_\alpha\}$ i.e. are such that $T(H_\alpha)$ is contained in H_α for each α. For such a T we define

$$p_\alpha(T) = ||T|_{H_\alpha}||$$

$L(H)$ is a Saks space with the uniform norm and the locally convex topology defined by the $\{p_\alpha\}$.

7 Saks spaces

In this section we consider special types of spaces with mixed topologies – those whose bornology is induced by a norm. We propose to call them Saks spaces since they coincide essentially with the spaces indtroduced under this name by ORLICZ (the precise relationship between these concepts is discussed in the notes). We are concerned here with basic constructions on Saks spaces. SInce these are based on the corresponding constructions on Banach spaces (which they generalise), we recall the latter briefly (see SEMADENI). The construction of subspaces and quotient spaces of normed spaces is well-known. If $\{(E_\alpha, || \, ||_\alpha)\}_{\alpha \in A}$ is a family of normed spaces, we define new normed spaces as follows: denote by E the Catersian product $\prod_{\alpha \in A} E_\alpha$ and define extend norms (i.e. taking on finite values)

$$\begin{aligned} || \, ||_1 : x = (x_\alpha) &\rightarrow \sum_{\alpha \in A} ||x_\alpha||_\alpha \\ || \, ||_\infty : x(x_\alpha) &\rightarrow \sup_{\alpha \in A} ||x_\alpha||_\alpha \end{aligned}$$

on E. Then if

$$\begin{aligned} E_1 &:= (x \in E : ||x||_1 < \infty\} \\ E_\infty &:= \{x \in E : ||x||_\infty < \infty\} \end{aligned}$$

$(E_1, || \, ||_1)$ and $(E_\infty, || \, ||_\infty)$ are normed spaces. They are Banach if (and only if) each E_α is a Banach space. We write $B \sum_{\alpha \in A} E_\alpha$ and $B \prod_{\alpha \in A} E_\alpha$ for E_1 and E_∞ resp. They satisfy the universal property that one exect of a sum and a product if we restrict attention to linear contractions.

If A is a directed set and

$$\{\pi_{\beta_\alpha} : E_\beta \rightarrow E_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}$$

(resp.

$$\{i_{\alpha\beta} : E_\alpha \rightarrow E_\beta, \alpha \leq \beta, \alpha, \beta \in A\})$$

is a projective spectrum (resp. an inductive spectrum) of normed spaces (i.e. each $\pi_{\beta\alpha}$ and each $i_{\alpha\beta}$ is a linear contraction with $\pi_{\alpha\alpha} = \mathrm{Id}_{E_\alpha}$ (resp. $i_{\alpha\alpha} = \mathrm{Id}_{E_\alpha}$) for each α and, if $\alpha \leq \beta \leq \gamma$, then $\pi_{\gamma\alpha} = \pi_{\beta\alpha} \circ \pi_{\gamma\beta}$ (resp. $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$)), then we define the projective limit of the first spectrum as the (closed) subspace

$$\{(x_\alpha) \in B \prod_{\alpha \in A} E_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for } \alpha \leq \beta\}$$

and denote it by $B - \lim_{\leftarrow} - \{E_\alpha, \pi_{\beta\alpha}\}$. Similarly, the inductive limit $B - \overset{\lim}{\longleftarrow}\{E_\alpha : i_{\alpha\beta}\}$ of the second spectrum is the quotient of $B \sum_{\alpha \in A} E_\alpha$ with respect to the closed subspace generated by the elements $(x_\gamma - i_{\beta\gamma}(x_\beta))$ $(\beta \leq \gamma)$ (we are regarding each space E_β as a subspace of $B \sum_{\alpha \in A} E_\alpha$ in the obvious way). In fact we shall only require the following special representation of an inductive limit: suppose that each E_α is a closed subspace of a given Banach space F and that A is so ordered that $\alpha \leq \beta$ if and only if $E_\alpha \subseteq E_\beta$ (and then $i_{\alpha\beta}$ is the natural injection): then the inductive limit is naturally identifiable with the closure of $\bigcup_{\alpha \in A} E_\alpha$ in F.

Lemma 5 *Let (E, τ) be a locally convex space, $||\,||$ a norm on E with unit ball $B_{|}|\,||$. Then the following are equivalent:*

(a) $B_{|}|\,||$ is τ-closed;

(b) $||\,||$ is lower semi-continuous for τ;

(c) $||\,|| = \sup\{p : p$ is a τ-continuous seminorm with $p \leq ||\,||\}$.

Proof. (c) $\Rightarrow$ (b) and (b) $\Rightarrow$ (a) follow immediately from the elementary properties of semi-continuous functions.

(a) $\Rightarrow$ (c): suppose $x \in E$ with $||x|| > 1$ i.e. $x \notin B_{|}|\,||$. We need only find a continuous seminorm p on E so that $p \leq ||\,||$ and $p(x) > 1$. By the Hahn-Banach theorem, there is an $f \in (E, \tau)'$ so that $|f| \leq 1$ on $B_{|}|\,||$ and $f(x) > 1$.

■

Definition 11 *A* **Saks space** *is a triple $(E, ||\,||, \tau)$ where E is a vector space, τ is a locally convex topology on E and $||\,||$ is a norm on E so that $B_{|}|\,||$, the unit ball of $(E, ||\,||)$, is τ-bounded and satisfies one of the conditions of 5. If $(E, ||\,||, \tau)$ and $(F, ||\,||_1, \tau_1)$ are Saks spaces, a* **morphism** *from E into F is a linear norm contraction from E into F so that $T?_{B_y}$ is $\tau - \tau_1$-continuous. A Saks space $(E, ||\,||, \tau)$ is* **complete** *if $B_{|}|\,||$ is τ-complete. Then $(E, ||\,||)$ is a Banach space 4.*

In constructing Saks spaces, one occasionally produces triples $(E, || \, ||, \tau)$ where all but the last condition (closedness of $B_{| \, ||}$) is satisfied. This forces us to take the following precaution: we define $B_1 := \bar{B}_{| \, ||}$ (closure in τ). Then the Minkowski funcitonal $|| \, ||_1$ of B_1 (as defined in 9) is a norm on E so that $(E, || \, ||_1, \tau)$ is a Saks space. The following Lemma ensures that we do not lose any morphism in this proces.

Lemma 6 *Let T be a linear mapping from E into a locall convex space F so that $T|_{B || \, ||}$ is τ-continuous. Then $T|_{B_1}$ is τ-continuous.*

PROOF. By 5, it suffices to show that $T|_{B_1}$ is continuous at zero. Let U be an absolutely convex neighbourhood of zero in F and choose an open neighbourhood V of zero in E so that $T(V \cap B_{| \, ||}) \subseteq 1/2U$. Then $T(V \cap V_1) \subseteq U$. For if $x \in V \cap B_1$ and we choose $\lambda \in]0, 1[$ so that $\lambda x \in B_{| \, ||}$, then we can, by the continuity of T in $B_{| \, ||}$, find $|| \, || \in V \cap B_{| \, ||}$ so that $T(\lambda x) - T(\lambda y) \in \lambda/2U$. Then

$$Tx = \lambda^{-1}(T(\lambda x) - T(\lambda y)) + Ty \in U.$$

■

7.1 The associated topology

If $(E, || \, ||, \tau)$ is a Saks space, we can form the mixed topology $\gamma[|| \, ||, \tau]$ – it is called the **associated locally convex topology** of E. Then a morphism between two Saks spaces is continuous for the associated topologies (27 and 6) and Saks space is a complete if and only if (E, γ) is a complete locally convex space (39). We repeat that, despite these facts (and others to follow), the relevant structure is that of a Saks space and not a locally convex space (this is one of the reasons that we have been careful not ot forget the norms in the definition of a morphism – thus a $|| \, ||$-continuous linear mapping need not be a morphism although it is, of course, a scalar multiple of one). Hence we shall stubbornly persist in defining notions like completeness, compactness etc. in terms of the structure as a Saks space even when these can be expresed in terms of $|| \, ||$ (using the theory of section 1).

7.2 Subspaces and quotient spaces

Let $(E, || \, ||, \tau)$ be a Saks space, F a vector subspace of E. Then if $|| \, ||_f, \tau_F$ denote the norm (resp. the locally convex topology) induced on F $(F, || \, ||_F, \tau_F)$ is a Saks space. There are examples which show that $\gamma[|| \, ||_F, \tau_F]$ need not coincide with $\gamma[|| \, ||, \tau]|_F$. If F is a γ-closed subspace, then we denote by

${}_F|| \,||$ and ${}_F\tau$ the structures induced on the quotient space E/F. The triple $(E/F, {}_F|| \,||, {}_F\tau)$ need not be a Saks space since it can happen that the unit ball of $(E/F, {}_F|| \,||, {}_F\tau)$ is not ${}_F\tau$-closed. However, by the process described before Lemma 6, we can obtain a Saks space which we shall call the **quotient Saks space**.

7.3 Completion

Let $(E, || \,||, \tau)$ be a Saks space and denote by $\hat{E}_\tau$ the complection of the locally convex space (E, τ). We write $\hat{B}$ for the closure of $B_{|| \,||}$ in $\hat{E}_\tau$ and $\hat{E}$ for the linear span of $\hat{B}$ in $\hat{E}_\tau$. Then if $|| \,||\hat{}$ is the Minkowski functional of $\hat{B}$ and $\hat{\tau}$ is the locally convex structure induced on $\hat{E}$ from $\hat{B}_\tau$, $(\hat{E}, || \,||\hat{}, \hat{\tau})$ is a complete Saks space. We call it the (**Saks space)-completion** of E. It has the following universal property: for every morphism T from $(E, || \,||, \tau)$ into a complete Saks space $(F, || \,||_m, \tau_1)$, there is a unique morphism $\hat{T}$ from $(\hat{E}, || \,||\hat{}, \hat{\tau})$ into $(F, || \,||_1, \tau_1)$ which extends T (for we can extend T firstly to $\hat{B}$ by uniform continuity and then to $\hat{E}$ by linearity).

As an example, consider the Saks space $(E, || \,||, \sigma(E, E'))$ where $(E, || \,||)$ is a normed space. The completion of $(E, \sigma(E, E'))$ is $(E')^*$ the algebraic dual of E'. The closure of B in (E') is its bipolar, i.e. the unit ball of E'', the bidual of $(E, || \,||)$. Hence the completion of $(E, || \,||, \sigma(E, E)')$ is E'', with the Saks space structure described in 2.A (as the dual of E'). THus we can regard the bidual of a normed space as a completion in the sense of Saks spaces.

7.4 Products and projective limits

Let $\{(E_\alpha, || \,||_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of Saks spaces. We can give $(E_\infty, || \,||_\infty)$, the normed space product of $\{E_\alpha\}$, a Saks space structure by considering on E_∞ the topology τ_∞¡ the product of $\{\tau_\alpha\}$. Then the unit ball $B_{|| \,||_\infty}$ of E_∞ is the product $\prod_{\alpha \in A} B_{|| \,||_\alpha}$ and so is τ_∞-closed. For the same reason, $(E_\infty, || \,||_\infty, \tau_\infty)$ is complete if and only if each $(E_\alpha, || \,||_\alpha, \tau_\alpha)$ is. We call $(E_\infty, || \,||_\infty, \tau_\infty)$ the **Saks space product** of $\{E_\alpha\}$ and denote it by $S \prod_{\alpha \in A} E_\alpha$.

If $\{\pi_{\beta\alpha} : E_\beta \to E_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}$ is a projective system of Saks spaces (so that the $\pi_{\alpha\beta}$'s are Saks space morphism), we can define its (Saks space) projective limit $(E, || \,||, \tau)$ as follows: as in the first paragraph of this section, we conside the space E of threads as a subspace of $S \prod_{\alpha \in A} E_\alpha$ and give it the induced structure in the sense of 7.2. It can easily by checked that the unit ball of E is τ_∞-closed in $S \prod_{\alpha \in A} E_\alpha$ and so E is complete if each E_α is.

We denote this projective limit by $S - \varprojlim\{E_\alpha, \pi_{\beta\alpha}\}$. Sums and inductive limits of Saks spaces can be defined without difficulty but we shall not require them.

We recall that each Banach space can be regarded as a Saks space – namely the Saks space $(E, || \, ||, \tau_{| \, ||})$. The following result shows that the Bananch spaces are in a certain sense dense in the Saks spaces and corresponds to the fact that complete locally convex spaces are projective limits of Banach spaces.

Proposition 40 *A Saks space $(E, || \, ||, \tau)$ is coplete if and only if it is the Saks space projective limit of a system of Banach spaces.*

PROOF. The sufficiency follows from the remarks above. Necessity: denote by S the family of τ-continuous seminorms p on E which are majorised by $|| \, ||$. Then S is a directed set with the natural (pointwise) ordering and $|| \, || = \sup S$ 5. If $p \in S$ we denote by $\hat{E}_p$ the Banach space associated with p (i.e. the completion of the space $E/N_[$ where $N_p := \{x \in E : p(x) = 0\}$, with the norm induced by p). If $p \leq q$, let pi_{pq} denote the natural contraction from $\hat{E}_q$ into $\hat{E}_p$. Then $\{\pi_{pq} : \hat{E}_q \to \hat{E}_p, p \leq q\}$ is a projection system of Banach spaces and it is not difficult to show that $(E, || \, ||, \tau)$ is its projective limit.

■

We remark that if $(E, || \, ||, \tau)$ is not complete, then the above construction produces its completion in the sense of 7.3

As an example of a Saks space product, consider a family $\{X_\alpha\}_{\alpha \in A}$ of locally compact spaces and let $\mathcal{S}_\alpha$ be a family of susts of X as in ?. Denote by X the topological sum of the spaces $\{X_\alpha\}$ and by $\mathcal{S}$ the family $\bigcup_{\alpha \in A} \mathcal{S}_\alpha$ (X_α is regarded as a subspace of X). Then the underlying vector space of $S \prod_{\alpha \in A} C^\infty(X_\alpha)$ can be naturally indentified with $C^\infty(X), || \, ||_\infty, \tau_{\mathcal{S}})$ and $S \prod_{\alpha \in A} C^\infty(X_\alpha)$. This example displays the suitability of a Saks space product in a situation where any locally covex product is hopelessly inadequate.

If X is a locally compact space, then

$$\{\rho_{K_1,K} : C(K_1) \to C(K), K, K_1 \in \mathcal{K}(\mathcal{X}), \mathcal{K} \subseteq \mathcal{K}_\infty)$$

(where $C(K)$ denotes the space of continuous, complex-valued functions on K and $rho_{K_1,K}$ is the restriction operator) is a projective spectrum of Banach spaces and its Saks space projective limit is $(C^\infty(X), || \, ||_\infty, \tau_{\mathcal{K}})$.

Saks spaces which are projective limits of finite dimensional spaces will play a role in the later chapters. They can be characterised as follows:

Proposition 41 *Let $(E, || ||, \tau)$ be a Saks space. Then the folllowing are equivalent:*

1. *B_y is τ-compact;*

2. *E is the Saks space projective limit of finite dimensional Banach spaces;*

3. *E has the form $(F', || ||, \sigma(F', F))$ for some Banach space.*

Then $\gamma = \tau_c(F', F)$, the topology of uniform convergence on the compact sets of E, is the **finest topology** *on E which agree with τ on $B_|| ||$. In fact, if 1) is fullfilled, then E is naturally identifiable with $S - \lim_{\overrightarrow{F \in \mathcal{F}(\mathcal{E}')}}$ where $\mathcal{F}(\mathcal{E}'|| ||)$ denotes the family of finite dimensional subspaces of E'_γ.*

Further, the following are equivalent:

1. *B is τ-compact and metrisable;*

2. *E is the Saks space projective limit of a sequence of finite dimensional Banach spaces;*

3. *E has the form $(F', || ||, \sigma(F', F))$ for a separable Banach space F;*

4. *$B_|| ||$ is τ-compact and normable (i.e. there is a norm $|| ||_1$ on E so that $\tau = \tau_{|| ||_1}$ on $B_|| ||$).*

Proof. It is clear that if E has a representation as a projective limit of finite dimensional space, then its unit ball is τ-compact. On the other hand, if the latter condition is fulfilled, then $(E, || ||)$ is semireflexive and so E is the dual of the Banach space E'_γ. It is then clear that E is the Saks space $(E, || ||, \sigma(E, E'_\gamma))$ and the rest of the proposition follows from standarddual-ity arguments.

∎

7.5 Duality

The dual of $(E, || ||, \tau)$ is defined to be the linear span of the set of morphism from E into $\mathbf{C}$ i.e. it is the dual of the locally convex space $(E, \gamma[|| ||, \tau])$ which is just E'_γ. It is a Banach space. Suppose now that E is the Saks space projective limit of the spectrum

$$\{\pi_{\beta\alpha} : E_\beta \to E_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}$$

of Saks space. We assume, in addition, that $\pi_\alpha(B_|| ||)$ is τ_α-dense in $B_{|| ||_\alpha}$ for each α (π_α is the natural morphism from E into E_α). This condition is satisfied, for example, by the canonical representation of E 40. Then each $(E_\alpha)'$ can be regarded as a Banach subspace of $(E, || \, ||)'$ and the natural injection $i_{\alpha\beta}$ from $(E_\alpha)'_\gamma$ into $E_\beta)'_\gamma$ $(a \leq \beta)$ is the transpose of $\pi_{\alpha\beta}$.

Proposition 42 *(a) E'_γ is the Banach space inductive limit of the spectrum*

$$\{i_{\alpha\beta} : (E_\alpha)'_\gamma \to (E_\beta)'_\gamma, \alpha \leq \beta, \alpha, \beta \in A\}.$$

(b) A subset H of $(E, || \, ||)'$ is γ-equicontinuous if and only if there is a sequence (α_n) with values in A and, foreach $n \in \mathbf{N}$, a subset H_n of $(E, || \, ||)'$ so that

(i) H_n is τ_{α_n}-equicontinuous;

(ii) $''sum_n \sup\{||f|| : f \in H_n\} < \infty$;

(iii) $H \subseteq \sum H_n$ (i.e. if $f \in H$, $f = \sum_{n=1}^\infty f_n$ where $f_n \in H_n$).

Proof.

(a) By a standard result on the duals of locally convex projective limits, the dual of (E, τ) is the subspace $\bigcup_{\alpha \in A}(E_\alpha, \tau_\alpha)'$ of $(E, || \, ||)'$. Hence by 35 (ii) and the remarks at the beginning of this section, E'_γ is the inductive limit of the Banach space $\{(E_\alpha)'_\gamma\}$.

b) We note firstly that if H' is a τ-equicontinuous subset of E' then there is an $\alpha \in A$ so that $H' \subseteq E'_\alpha$ and H' is τ_α-equicontinuous. The sufficiency of the condition follows then from 37. On tyeh other hand, if H is γ-equicontinuous then there are α_0, α_1 in $A(\alpha_0 < \alpha_1)$ and $H_0 \tau_{\alpha_0}$-equicontinuous in E' (resp. H_1 τ_{α_1}-equicontinuous is E') so that

$$H \subseteq H_0 + \frac{\epsilon}{2}B, \quad H \subseteq H + \frac{\epsilon}{2}B$$

($\epsilon > 1$, B the unit ball of E'). Then if $h \in H$, it has a representations

$$h_0 + \frac{\epsilon}{2}b_0; h_1 + \frac{\epsilon}{2}b_1 \quad (h_0 \in H_0, h_1 \in H_1, b_0, b_1 \in B).$$

Then $h = h_0 + (h_1 - h_0) + \frac{\epsilon}{2}b_0$ and $||h - 1 - H_0|| \leq \epsilon$. We define H_1 to be the set of all such $(h_0 - h_1)$ required in the representations of the elements of H. Continuing inductively, we can construct a sequence (H_n) with the required properties.

■

7.6 Co-Saks space

It will be convenient to introduce an intrinsic definition of those spaces which are duals of Saks spaces. We define a **Co-Saks space** to be a vector space $\mathcal{E}$ with the following structures:

a) a norm $|| \, ||$;

- a convex bornology $\mathcal{B}$ consisting of $|| \, ||$-bounded sets;

c) a locally convex topology σ on E for which $B_{|| \, ||}$ is closed and each $B \in \mathcal{B}$ is relatively compact.

   In addition, we assume that the following compatibility condition holds.

d) if $C \subseteq \mathcal{E}$ is an absolutely convex set so that for each $\epsilon > 0$, there is a $B \in \mathcal{B}$ with $C \subseteq B + \epsilon B_{|| \, ||}$, then $C \in \mathcal{B}$.

The following are examples of Co-Saks spaces:

I. The dual E'_γ of a Saks spae is a Co-Saks space when we choose for $\mathcal{B}$ the family of γ-equicontinuous subsets and for σ the topology $\sigma(E'_\gamma, E)$.

II. If S is a completely regular space, we define a Co-Saks structure on $C^\infty(S)$ as follows:

   $|| \, ||$ is the supremum norm;
   $\mathcal{B}$ is the family of uniformly bounded, equicontinuous subsets of $C^\infty(S)$;
   σ is the topology of pointwise convergence on S.

III. We can generalise II as follows: let S be a uniform space and denote by $\mathcal{U}^\infty(\mathcal{S})$ the space of bounded, uniformly continuous functions on S. We can define a Co-Saks structure on S as above (replacing "equicontinuous" by "uniformly equicontinuous"). If S is a completely regular space and we ragard it as a uniform space with the **fine** uniformity (i.e. the finest uniform structure compatible with the topology), then $C^\infty(S) = \mathcal{U}^\infty(\mathcal{S})$ and the Co-Saks structures defined in II and III coincide.

7.7 Duality for Co-Saks spaces

If $(\mathcal{E}, || \, ||, \mathcal{B}, \sigma)$ and $(\mathcal{F}, || \, ||_\infty, \mathcal{B}_\infty, \sigma_\infty)$ are Co-Saks spaces then the linear mapping $T : \mathcal{E} \to \mathcal{F}$ is a (Co-Saks)-morphism if

a) it is $|| \, || - || \, ||_1$ bounded;

b) it is $\mathcal{B} - \mathcal{B}_\infty$ bounded;

c) the restrictions of T to sets $B \in \mathcal{B}$ are $\sigma - \sigma_1$ continuous.

This is equivalent to the fact hat T is $\tilde{\sigma} - \tilde{\sigma}_1$ continuous where $\tilde{\sigma}$ is the finest locally convex topology on $\mathcal{E}$ which coincides with σ on the set of $\mathcal{B}$ – and $\tilde{\sigma}_1$ is defined on $\mathcal{F}$ in the corresponding way.

The dual $\mathcal{E}'$ of a Co-Saks space is the space of all morphisms $f : \mathcal{E} \to \mathbf{C}$. It has a natural Saks space structure $(\mathcal{E}', || \, ||, \tau)$ where y is the norm dual to y on $\mathcal{E}$ and τ is the topology of uniform convergence on the sets of $\mathcal{B}$. The definition of the norm on $\mathcal{E}'$ is justified by the following Lemma:

Lemma 7 *Let f be a linear functional on the Co-Saks spece $(\mathcal{E}, || \, ||, \mathcal{B})$. Then f is norm-bounded (and so an element of $\mathcal{E}'$) provided $f|_B$ is $\sigma|_B$-continuous for each $B \in \mathcal{B}$.*

PROOF. If f were not bounded on $B_{| \, ||}$, we could find a sequence (x_n) in $B_{| \, ||}$ so that $|f(x_n)| \gamma n^2$. Now $\{x_n/n\}$ is in $\mathcal{B}$ by 7.6 d) and hence f is bounded on this set – contradiction.

■

It follows now from standard duality theory for locally convex space that there is a complete duality between Co-Saks spaces and complete Saks space. The dual of one type of space is a member of the dual class and the natural mapping of a Saks (Co-Saks) space into its bidual is an isomorphism. The following result will be useful later when we study uniform measures.

Proposition 43 *Let $(E, || \, ||, \tau)$ be a complete Saks space, $\gamma = \gamma[|| \, ||, \tau]$ the associated mixed topology. Consider the dual Co-Saks space $(\mathcal{E}, || \, ||, \tilde{\mathcal{B}}, \sigma)$ and write $\mathcal{B}$ for the bornology of τ-equicontinuous sets on $\mathcal{E}$, respectively $\tilde{\sigma}$ for the finest locally convex topology on $\mathcal{E}$, which agrees with σ on the set of $\tilde{B}$. Then the following conditions are equivalent for a subset H of $\mathcal{E}$:*

a) H is $\tilde{\sigma}$-equicontinuous;

b) H is $\tilde{\sigma}$-equicontinuous on the sets of $\tilde{B}$;

c) H is norm-bounded and $\tilde{\sigma}$-equicontinuous on the sets of $\mathcal{B}$;

d) H relatively γ-compact (resp. γ-precompact);

e) H is norm-bounded and τ-precompact.

This follows from previous results, standard duality theory and Ascoli's theorem on relatively compactness in $C(K)$-spaces.

7.8 The Hom functor

If E is a Banach space and $(F, || \, ||, \tau)$ is a Saks space, then we denote by Hom (E, F) the set of $|| \, ||\gamma$ continuous linear operators from E into F. Note that as a vector space, this coincides with the space of norm-bounded linear operators from E into F. We regard Hom (E, F) as a Saks space with the supremum norm and the topology τ_p of pointwise convergence (with respect to τ) on E. Note that on the unit ball of Hom (E, F), this topology coincides with that of compact convergence, resp. of pointwise convergence on a dense subspace of E. The next result is an easy consequence:

Proposition 44 *1. If $\{E_\alpha\}$ is an inductive system of Banach spaces and F is a complete Saks space, then there is a natural isomorphism between the Saks spaces*

$$Hom(B - \alpha \text{ and } S - \lim_{\longleftarrow \alpha} Hom(E_\alpha, F).$$

In particular, if E is a Banach space

$$Hom(E, F) = Hom((B - \lim_{\overrightarrow{G \epsilon \mathcal{F}(\mathcal{E})}} G), F) = S \lim_{\overrightarrow{G \epsilon \mathcal{F}(\mathcal{E})}})G, F)$$

($\mathcal{F}(\mathcal{E})$ is the directed family of finite-dimensional subspaces of E).

2. If $\{F_\alpha\}$ is a projective system of Banach spaces, E a Banach space, then there is a natural isomorphisms between the Saks spaces

$$Hom(E, F) = Hom(E, S - \lim_{\overrightarrow{G \epsilon \mathcal{F}(\mathcal{F}'_\gamma)}} G') = S - \lim_{\overrightarrow{G \epsilon \mathcal{F}(\mathcal{E})}} Hom(E, G').$$

7.9 Tensor products

Let $(E, || \, ||, \tau)$ and $(F, || \, ||_1, \tau_1)$ be Saks spaces. We denote by $E \otimes F$ the algebraic tensor product of E and F. On $E \otimes F$ we consider the following structures:

$|| \, ||_\otimes$: $x \to \sup\{|(f \otimes g)(x)| : f\mathcal{B}_{\mathcal{E}'_\tau}, \}\mathcal{B}_{\mathcal{F}'_{\tau\infty}}\}$

$\tau \hat{\hat{\otimes}} \tau_1$: the projective tensor product of τ and τ_1;

$\tau \hat{\hat{\otimes}} \tau_1$: the inductive tensor product of τ and τ_1.

Then $(E \otimes F, || \, ||_\otimes, \tau \hat{\otimes} \tau - 1)$ and $(E \otimes F, || \, ||_\otimes, \tau \hat{\hat{\otimes}} \tau_1)$ are Saks spaces (this follows from 5 since $|f \otimes g|$ is continuous for $\tau \hat{\hat{\otimes}} \tau - 1$ and $\tau \hat{\hat{\otimes}} \tau - 1$. We denote their completions by projective limits, we can give another construction of the tensor product $E \hat{\hat{\otimes}}_\gamma F$: let

$$\pi_{qp} : \hat{E}_q \to \hat{E}_p : p, q \epsilon S, p \leq q\}$$

and

$$\pi_{q_1 p_1} : \hat{E}_{q_1} \to \hat{E}_{p_1} : p_1, q_1 \epsilon S_1, p_1 \leq q_1\}$$

be canonical representations of the completions of E and F. Then

$$\{\pi_{qp} \otimes \pi_{q_1 p_1} : E_q \hat{\hat{\otimes}} F_{q_1} \to E_p \hat{\hat{\otimes}} E_{p_1}; p \leq q, p_1 \leq q_1\}$$

is a projective spectrum of Banach spaces and its projective limit is naturally isomorphic to $(E \hat{\hat{\otimes}}_\gamma F, || \, ||, \tau \hat{\hat{\otimes}} \tau_2)$.

We consider some results on operators between Saks spaces.

Proposition 45 *Let $(E, || \, ||, \tau)$ and $(F, || \, ||_1, \tau - 1)$ be Saks spaces, whereby $(F, || \, ||_1)$ is a Banach space. Then a linear operator $T : E \to F$ maps a γ-neighbourhood of zero in E into a relatively (weakly) compact subset of (F, γ) if and only if T maps bounded sets into relatively (weakly) compact sets and is $|| \, || - y$ continuous.*

Proof. The condition is clearly necessary. Suppose then that there is an absolutely convex γ-neighbourhood U of zero so that $T(U) \subset B_F$. Hence if $B_k(E) := kB_E$, we define

$$\tilde{U} = bigcup_{n=1} \sum_{k=1}^{n} (2^{-k} U \cap B_k(E)).$$

Then $\tilde{U}$ is a γ-neighbourhood of zero and its immage under T is relatively (weakly) compact. For

$$\begin{aligned} T(\tilde{)} &= \bigcup_{n=1} \sum_{k=1}^{n} T(2^{-k} U \cap B_K(E)) \\ &\subset \sum_{k=1}^{n} (T(2^{-k} U \cap B_K(E)) + \bigcup_{k=n+1}^{l} T(2^{-k} U) \\ &\subset \sum_{k=1}^{n} T(2^{-k} U \cap B_K(E)) + B_2 - n(F) \end{aligned}$$

for each n. Since the first term is relatively (weakly) compact,the result follows from a well-known criterium for weak compactness resp. its analogue for norm compactness.

∎

At this point we recall a result on the factorisation of weakly compact operators:

Proposition 46 *Let E and F be Banach spaces, $T : E \to F$ a weakly cmpact linear operator. Then there are a reflexive Banach space G and contiuous linear operators $S : G \to F$ and $R : E \to G$ so that $T = SR$. If T is compact, we can find G, R, S with the additional properties that G is separable and R and S are compact.*

This follows from the following lemma:

Lemma 8 *Let E be a Banach space and W a weakly compact, absolutely convex subsets. Then there exists a weakly compact, absolutely convex subset C of E which contains W and is such the associated normed space $(E_C, || \, ||_C)$ is reflexive. If W is norm compact, then one may construct C so that W compact in $(E_C, || \, ||_C)$, C is compact in E and $(E_C, || \, ||_C)$ is a separable, reflexive Banach space.*

PROOF. Denote by B the unit ball of E and put $W_n := 2^n K + n^{-1} B$. Then W_n is a closed, absolutely convex subset of E¿ Let $|| \, ||_n$ be the Minkowski funcitonal of W_n and write E_n for E, regarded as a normed space with this norm. Since it is equivalent to the original norm, E_n is a Banach space. Now if

$$C := \left\{ x \in E : \sum_{n=1}^{\infty} ||x||_n^2 \leq 1 \right\},$$

then C is a closed, absolutely convex subset of E and a simple calculation shows that K is contained in C while the latter si a subet of $2^n K + n^{-1} B$ for each n and so is weakly compact by a standard characterisation of weak compactness. We show that $\sigma(E, E')$ and $\sigma(E_C, E'_C)$ coincide on C and this will conclude the first part. Note that the diagonal mapping $x \to (x, x, \dots)$ is an isometric embedding from E_C onto a closed subspace of the ℓ^2-sum

$$\ell^2(E_n) = \left\{ x = (x_n) \in \prod_{n=1}^{\infty} E_n : \sum_n ||x_n||_n^2 < \infty \right\}.$$

The dual of $\ell^2(E_n)$ is $\ell^2(E'_n)$ (proof exactly as for the duality of ℓ^2) and so it suffices to show that $\sigma(\ell^2(E_n), \ell^2(E'_n))$ coincides with $\sigma(E, E')$ on C. But the former agreees ont the bounded set C with the weak topology induced by the dence subspace of $\ell^2(E'_n)$ consisting of the sequence with only finitely many non-zero elements which implies the result.

■

Now we turn to the second part. If K is a norm-compact, then so is C since it is clearly totally bounded.

We now show that K is compact in E_C. It will suffice to show that it is precompact. Let $\epsilon > 0$ – we shall find an ϵ-net for K with respect to $|| \, ||_C$. First note that if $x \epsilon D$ then $||x_n| \leq 2^{-n}$. Hence there is an $N > 0$ so that $(\sum_{n=N+1}^{\infty} ||x||_n^2) \leq \epsilon^2/8$ for each $x \epsilon K$. Now the norm $(\sum_{n=1}^{N} || \cdot ||_n^2)^{1/2}$ is equivalent to $|| \, ||$ and so there is a finite set $\{x_1, \dots, x_k\}$ in K so that for each $x \epsilon K$ there is an i with $(\sum_{n=1}^{N} ||x - x_i||_n^2) \leq \epsilon^2/2$.

Then we have $||x - x_i||_C \leq \epsilon$.

To show that $(E_C, || \, ||_C)$ is separable, note that since C is norm compact, the nrom topology agrees with $\sigma(E, E')$ on C and so the latter is metrisable. However, as we know, $\sigma(E, E;)$ agrees with $\sigma(E_C, E'_C)$ on C and so the latter is also metrisable. From this it follows that $(E_C, || \, ||_C)$ is separable.

Proposition 47 *Let $(E, || \, ||, \tau)$ be a Saks space, $(F, \tilde{\tau})$ a Locally convex space, $T : E \to F$ γ-continuous and linear. Then*

1. *T takes bounded sets into relatively compact sets if and only if T factorises as follows*

   where $(G, || \, ||_1, \tau_1)$ is a Saks space with τ_1-compact unit ball $R : E \to G$ is $\gamma - \gamma$-continuous (and so takes bounded sets into relatively compact sets) and $S : G \to F$ is $\gamma - \tilde{\tau}$ continuous. If F is a Banach space with the weak topology, we can assume that $(G, || \, ||_1)$ is reflexive and $\tau - 1 = \sigma(G, G')$ while if it is a Banach space with the norm topology, we can assume that $(G, || \, ||_1)$ is a separable and reflexive, with $\tau - 1 = \sigma(G, G')$ (so that the unit ball of G is $\tau - 1$-compact and metrisable).

2. *T is γ-compact if and only if it has a factorisation as in the diagram whereby R is $\gamma - || \, ||_1$ continuous.*

Proof. The necessity of the condition is clear. We prove the sufficiency. Let C be the closure of the image of the unit ball under T. Then

$$(E_C, || \, ||_C, \tau|_{E_C})$$

is the required space. S is simply T regarded as a mapping into F_C and S is the injection from F_C into F.

Now if F is a Banach space with the weak topology, then T is weakly compact as a mapping from the normed space E into the normed space F. Hence by the factorisation theorem, there is a ball B_1 which contains $T(B)$ in F and is such that the corresponding normed space E_{B_1} is reflexive. Then if we put $G := E_{B_1}$, τ and $\sigma(G, G')$ coincide on B_1 by compactness and so we can complete the proof as above.

If F is a Banach space with the norm topology, then we can proceed as above but, using the factorisation theorem for compact operators, we can assume that B_1 is norm compact.

2. $\Leftarrow$ is clear for then R is γ-compact by 45 and hence so is T.

$\Rightarrow$: Suppose that T is γ-compact. Then there is an absolutely convex γ-neigbourhood V of zero so that $C_1 = \overline{T(V)}$ is $\tilde{\tau}$-compact. We can construct G as above, using C_1 as the unit ball. Then R is γ-compact as it sends V into a compact set in (G, γ).

■

We finish this section with some brief comments on spaces which generalise the class of Banach algebras (resp. C^*-algebras) exactly as the Saks spaces generalise the class of Banach spaces.

Definition 12 *Let A be an algebra with unit e. A* **submultiplicative seminorm** *on A is a seminorm p with the properties $p(xy) \leq p(x)p(y)$ $(x, y \in A)$ and $p(e) = 1$. If A has an involution $x \to x^*$, p is a C^*-***seminorm** *if, in addition, it satisfies the condition $p(X^*x) = \{p(x)\}^2$ $(x \in A)$. A* **Saks algebra** *is a triple $(A, ||\,||, \tau)$ where $(A, ||\,||)$ is a Banach algebra with unit and the locally convex topology can ve defined by a family S of submultiplicative seminorms so that $||\,|| = \sup S$. A* **Saks C^*-algebra** *is defined in exactly the same way with the additional requirements that A have an involution and the seminorms of S be C^*-seminorms. It follows from this condition that $(A, ||\,||)$ is then a C^*-algebra.*

If p is a submultiplicative seminorm, then $\hat{A}_p$ (as defined in the proof of 40 has a natural Banach algebra structure. Hence a Saks algebra A has a canonical representation as a projective limit of spectrum

$$\{\pi_{qp} : \hat{A}_q \to \hat{A}_p, p \leq q, p, q \in S\}$$

where each $\hat{A}_p$ is a Banach algebra and the linking mappings are unit-preserving homomorhisms. Similarly, a Saks C^*-algebra has a representation with C^*-algebras as components and C^*-algebra homomorhisms as linking mappings.

7.10 The spectrum

If $(A, ||\,||, \tau)$ is a Saks algebra, we denote by $M_\gamma(A)$ the set of γ-continuous multiplicative functionals f from A into $\mathbf{C}$ with $f(e) = 1$. $M_\gamma A$ is called the **spectrum** of A. We regard $M_\gamma(A)$ as a topological space with the weak topology induced from $(A', \sigma(A', A))$. Then $M_\gamma(A)$, a subspace of a locally convex space, is completely regular. $M_\gamma(A)$ is a (topological) subspace of the spectrum of the Banach algebra $(A, || \, ||)$.

8 The space $C^\infty(S)$

We now consider the most developed field of applications of mixed topologies – the theory of strict topologies on spaces of bounded, continuous functions. In fact, BUCK's original work on such topologies was one of the moving factor in the development of the theory. In our treatment we have tried to emphasise the unifications and simplifications which can be achieved by presenting the theory from the point of view of the topics of chapter ??.

In the next section we consider the basic properties of strict topologies, in particular establishing the identity of the mixed topology $\gamma[|| \, ||, \tau_k]$ with Buck's topology which was defined a very general Stone-Weierstrass theorem, characterisations of separability of $C^\infty(X)$ and a result on its Mackey topology.

In §2 we study the algebraic structure of $C^\infty(X)$, identigying its spectrum and obtaining a Gelfand-Naimark type theory with its consequences for such spaces. §3 is dedicated to duality theory for $C^\infty(X)$ – in particular, with the determination of its dual as the space of bounded Radon measures on X and some results on the γ-equicontinuous subsets thereof (these coincide with the classical notion of uniformly tight sets of measures). The fourth? section is devoted to spaces of vector-valued continuous functions and in the fifth we define generalised strict topologies on $C^\infty(X)$ which are related to various continuity properties for measures and have received some attention recently.

The six? section is devoted to representation theorems for operators on $C^\infty(X)$-spaces. Using the apparatus of chapter I, we obtain simple and trasparent proofs of results which subsume and generalise results on operators from $C(K)$-spaces into Banach spaces. In the final section, we give a brief survey of topic of uniform measures. Here the structure of a Co-Saks space plays a central role. The main result is a compact ness theorem of PACHL, for which we bring a simplified proof.

9 The strict topologies

In this chapter, X will always denote a completely regular Hausdorff topological space, $\mathcal{S}$ a **saturated family of closed subsets** of X (that is, $\cup\mathcal{S}$ is dense n X and $\mathcal{S}$ is closed under the formation of finite unions and of closed subsets). Such a family is **of countable type** if there is a countable subfamily $\mathcal{S}_\infty$ so that each $B \in \mathcal{S}$ is contained in some $B_1 \in \mathcal{S}$.

Examples of saturated families are:

$\mathcal{F}$: the finite subsets of X;

$\mathcal{K}$: the compact subsets of X;

$\mathcal{B}$: the bounded, closed subsets of X (recall that a subset $B \subseteq X$ is bounded if each continuous, real-valued function on X is bounded on B).

We denote by

$C(X)$ – the space of continuous, complex-valued functions on X.

$C^\infty(X)$ – the space of bounded, continuous, complex-valued functions on X.

$|| \, ||_\infty$ denotes the supremum norm on $C^\infty(X)$. If $B \subseteq X$ then

$$\{p_B : x \to \sup\{|x(t)| : t \in B\}$$

is a seminorm no $C^\infty(X)$. If B is bounded, we can regard it as a seminorm on $C(X)$. If $\mathcal{S}$ is a saturated family, then $\tau_\mathcal{S}$ denotes the locally convex topology defined by the seminorms $\{p_B : B \in \mathcal{S}\}$. Then $(C^\infty(X), || \, ||_\infty . \tau_\mathcal{S})$ is a Saks space and we denote by $\beta_\mathcal{S}$ the associated mixed topology.

Proposition 48 *1. $\tau_\mathcal{S} \subseteq \beta_\mathcal{S} \subseteq \tau_{|| \, ||_\infty}$ and $\tau_\mathcal{S} - \beta_\mathcal{S}$ on the norm bounded sets;*

2. a subset of $C^\infty(X)$ is $\beta_\mathcal{S}$-bounded if and only if it is not bounded;

3. a sequence (x_n) in $C^\infty(X)$ is $\beta_\mathcal{S}$-convergent to zero if and only if it is norm bounded and $\tau_\mathcal{S}$-convergent to zero;

4. a subset of $C^\infty(X)$ is relatively $\beta_\mathcal{S}$-compact if and only if it is norm bounded and relatively $\tau_\mathcal{S}$-compact;

5. $(C^\infty(X), \beta_\mathcal{S})$ is barrelled or bornological if and only if $\mathcal{B}_\mathcal{S} = \tau_{|| \, ||_\infty}$;

6. a linear operator T from $C^\infty(X)$ into a locally convex space is $\beta_\mathcal{S}$-continuous if and only if its restriction to the unit ball of $C^\infty(X)$ is $\tau_\mathcal{S}$-continuous;

7. if $\mathcal{S}$ is of countable type, then the unit ball of $C^\infty(X)$ is $\tau_\mathcal{S}$-metrisable and so linear mapping T from $C^\infty(X)$ into a locally convex space is $\beta_\mathcal{S}$-continuous if and only if it is sequentially continuous.

Corollar 13 *A subset of $C^\infty(X)$ is $\beta_\mathcal{K}$-precompact if and only if it is norm bounded and equicontinuous.*

Definition 13 *Let $\mathcal{S}_\infty$ and $\mathcal{S}_\in$ be saturated families in X. X is $\mathcal{S}_\infty - \mathcal{S}_\in$* **normal** *if for every pair A, B of disjoint, closed subsets of X with $A \in \mathcal{S}_\infty$, $B \in \mathcal{S}_\in$ there is a continuous function $x : x \to [0,1]$ with*

$$x|_A = 0 \text{ and } x|_B = 1.$$

We say that X is **$\mathcal{S}$-normal** *if X is $\mathcal{S} - \mathcal{P}$ normal where $\mathcal{P}$ is the family of all closed subsets of X.*

Note that every X is $\mathcal{K}$-normal (in fact, for a Hausdorff space, $\mathcal{K}$-normality is equivalent to complete regularity – see BUCHWALTER.

The following Proposition is a generalisaton of URYSOHN's theorem and can be proved in exactly the same way.

Proposition 49 *X is $\mathcal{S} - \mathcal{S}$ normal if and only if for each $A \in \mathcal{S}$ and each continuous mapping $x : A \to [0,1]$ there is a continuous $\tilde{x} : x \to [0,1]$ so that $\tilde{x}|_A = x$.*

Proposition 50 *The sufficiency of the condition $X \in \mathcal{S}$ is trivial. Suppose that $X \notin \mathcal{S}$. Then we can find, for each seminorm p_B ($B \in \mathcal{S}$), an X in the unit ball of $C^\infty(X)$ so that $p_B(x) = 0$ and $||x||_\infty = 1$ which implies the result.*

In the following Proposition, we examine the problem of the completeness of $(C^\infty(X), \beta_\mathcal{S})$. It is convenient to introduce the following concept: a space X is **$\mathcal{S}$-complete** if each mapping $x : X \to \mathbf{C}$ is continuous if and only if $x|_A$ is continuous for each $A \in \mathcal{S}$ (it suffices to consider bounded functions). For example, X is $\mathcal{F}$-complete if and only if it is discrete. The $\mathcal{K}$-complete spaces are precisely the k_R-spaces.

Locally compact spaces and metrisable spaces are $\mathcal{K}$-complete. To each space X one can associate an $\mathcal{S}$-complete space in natural way: we give X the weak topology defined by the family of mappings from X into $\mathbf{C}$ which are such that their restitions to each $A \in \mathcal{S}$ are continuous. We denote X with this topology by $X_\mathcal{S}$. Then $X_\mathcal{S}$ is $\mathcal{S}$-complete and X is $\mathcal{S}$-complete if and only if $X = X_\mathcal{S}$. $C^\infty(X_\mathcal{S})$ is precisely the space of bounded mappings from X into $\mathbf{C}$ whose restrictions to teh sets of $\mathcal{S}$ are continuous.

Proposition 51 *Suppose that $\cup\mathcal{S} = \mathcal{X}$. Then $C^\infty(X), \beta_\mathcal{S})$ is complete if X is $\mathcal{S}$-complete. The converse is true if X is $\mathcal{S} - \mathcal{S}$-normal.*

PROOF. Suppose that X is $\mathcal{S}$-complete. Let $(x_\alpha)_{\alpha\in I}$ be a $\tau_{\mathcal{S}}$-Cauchy net in $B_{||\,||_\infty}$, the unit ball of $C^\infty(X)$. Then (x_α) converges pointwise to a function x from X into $\mathbf{C}$ and to restriction of x to $A \in \mathcal{S}$ is continuous, as the uniform limit of $(x_{\alpha|_{A_{\alpha\in I}}}$. Hence x is continuous and $x_\alpha \to x$ in ?

■

Now suppose that $(C^\infty(X), \beta_{\mathcal{S}})$ is complete and that $x : X \to \mathbf{C}$ is such that $x|_A$ is continuous for each $A \in \mathcal{S}$. We show that x is continuous when X is $\mathcal{S} - \mathcal{S}$ normal. It is no loss of generality to suppose that $x : X \to [0,1]$. For each $A \in \mathcal{S}$, let x_A be a continuous function from X into $[0,1]$ so that $x_A = x$ on A 49. Then $(x_A)_{A\in\mathcal{S}}$ is Cauchy net for $\beta_{\mathcal{S}}$ and so converges to a function in $C^\infty(X)$ i.e. $x \in C^\infty(X)$.

Corollar 14 *1. $(C^\infty(X)\beta_{\mathcal{K}})$ is complete if and only if X is a k_R-space;*

2. $(C^\infty(X)\beta_{\mathcal{F}})$ is complete if and only if X is discrete.

Corollar 15 *Suppose that X is $\mathcal{S} - \mathcal{S}$ normal and that $\cup\mathcal{S} = \mathcal{X}$. Then the completion of $(C^\infty(X), \beta_{\mathcal{S}})$ is $(C^\infty(_{\mathcal{S}}), \beta_{\mathcal{S}})$.*

These results can be intepreted as follows: under the restriction operators, the family $\{C^\infty(A); A \in \mathcal{S})$ of Banach space projective limit of this spectrum is naturally identificable with $(C^\infty(X_{\mathcal{S}}), ||\,||_\infty, \tau_{\mathcal{S}})$, in particular, with $(C^\infty(X), ||\,||_\infty, \tau_{\mathcal{S}})$ if X is $\mathcal{S}$-complete.

Now we give a concrete representation of a family of seminorms which defines $\beta_{\mathcal{S}}$. From these one can easily deduce that the mixed topology $\beta_{\mathcal{S}}$ reduces, in special cases, to the strict topologies which have been studied on space of bounded, continuous functions. We denote by $L^+_{\mathcal{S}}$ the set of bounded non negative upper semi-continuous funcitons ϕ on X which **vanish at infinity with respect to $\mathcal{S}$** i.e. which satisfy the condition: for each $\epsilon > 0$,

$$\{t \in X''\phi(t) \geq \epsilon\} \in \mathcal{S}.$$

If $\phi \in L^+_{\mathcal{S}}$,

$$p_\phi : x \to ||\phi x||_\infty$$

is a seminorm on $C^\infty(X)$. The family of all such seminorms defines a locally convex topology $\tilde{\beta}$ on $C^\infty(X)$ (note that the characteristic function of each $A \in \mathcal{S}$ is in $L^+_{\mathcal{S}}$ and so $\tilde{\beta}$ is finer that $\tau_{\mathcal{S}}$).

Proposition 52 *If X is $\mathcal{S} - \mathcal{S}$ normal, then $\tilde{\beta} = \beta_{\mathcal{S}}$.*

Proof. We note first that $(C^\infty(X), || ||_\infty, \tau_\mathcal{S})$ satisfies condition a) of I.4.4.(ist nicht da). If $A \in \mathcal{S}$ and $x \in C^\infty(X)$ then, by 49, there is a $y \in C^\infty(X)$ so that $y = x$ on A and so $x = y + z$ is a suitable decomposition of x.

■

We now show that $\beta_\mathcal{S}$ is finer that $\tilde{\beta}$ i.e. that $\tilde{\beta}$ is coarser than $\tau_\mathcal{S}$ on B_{y_∞}. If $\phi \in L^+_\mathcal{S}$, $\epsilon > 0$ and $A := \{t \in X : \phi(t) \geq \epsilon\}$ then for $x \in B_{|| ||_\infty}$ with $p_A(x) \leq \{\epsilon \sup_{t\in X} |\phi(t)|\}^{-1}$ we have $p_\phi(x) \leq \epsilon$. On the other hand if V is a $\beta_\mathcal{S}$-neighbourhood of zero, then by I.4.4., it contains a set of the form

$$\{x \in C^\infty(X) : p_{A_n}(x) \leq l_n\}$$

where (A_n) is an increasing sequence in $\mathcal{S}$ and (l_n) is a strictly increasing sequence of positive numbers which converge to infinity. Then if

$$\begin{array}{rcl} & & l_1^{-1} \quad (t \in A_1) \\ \phi : t & \to & l_n^{-1} \quad (t \in A_n \setminus A_{n-1}) \\ & & 0 \quad (t \in X \setminus \cup A_n) \end{array}$$

ϕ is in $L^+_\mathcal{S}$ and V contains the unit ball of p_ϕ¿

We now give a Stone-Weierstraß theorem for $\mathcal{B}_\mathcal{S}$. For convenience, we consider the space $C^\infty_\mathbf{R}(X)$ of real valued funcitons on X as a real vesctor space. The results can be extended to complex-valued funcitons using standard methods. Since the sets of $\mathcal{S}$ are not necessarily compact, we need a refinement of the classical Stone-Weierstraß theorem due to NEL. Recall that a subset of X is a **zero-set** if it has the form $x^{-1}(0)$ for some $x \in C^\infty_\mathbf{R}(X)$. A subset M of $C^\infty_\mathbf{R}(X)$ **separates disjoint zero-sets** if for each pair A, B of disjoint zero sets in X, there is an $x \in M$ so that $\overline{x(A)}$ and $\overline{x(B)}$ are disjoint. The following Lemma follows from the fact that the points of the Stone-Čech compactification βX of X are limits of z-ultrafilters in X. Its proof can be found in NEL.

Lemma 9 *Let M be a subset of $C^\infty_\mathbf{R}(X)$ which separates disjoint zero sets in X. Then M, regarded as a subset of $C^\infty_\mathbf{R}(\beta X)$, separates the points of βX.*

Proposition 53 *Suppose that M is a subalgebra of $C^\infty_\mathbf{R}(X)$ so that for each $A \in \mathcal{S}$, M_A, the restriction of M to A, separates disjoint zero-sets in A and contains a function which is bounded away from zero on A. Then M is $\beta_\mathcal{S}$-dense in $C^\infty_\mathbf{R}(X)$.*

Proof. We can assume that M is $\beta_{\mathcal{S}}$-closed. Then it is norm closed and so is a lattice under the pointwise ordering. We show that if $x \in C^{\infty}_{\mathbf{R}}(X)$, $0 < \epsilon < 1$ and if $A \in \mathcal{S}$, then there is a $y \in M$ so that $||y||_{\infty} \leq ||x||_{\infty} + 1$ and $p_A(x - y) \leq \epsilon$. M_A, regarded as an algebra of functions on βA, satisfies the conditions of the classical Stone-Weierstraß theorem (for M_A separates the points of βA by 9. Hence M_A is norm-de? in $C^{\infty}_{\mathbf{R}}(A)$ and so there isa $y_1 \in M$ with $p_A(x - y)_1) \leq \epsilon$. The

$$y := \sup\{\inf(y_1, ||x||_{\infty} + 1), -(||x||_{\infty} + 1)\}$$

is the required functiton.

■

Corollar 16 *Let M be a subalgebra of $C^{\infty}_{\mathbf{R}}(X)$ so that separates the points of X and for each $t \in X$ there is an $x \in M$ with $x(t) \neq 0$. Then M is $\beta_{\mathcal{K}}$-dense in $C^{\infty}_{\mathbf{R}}(X)$.*

We now consider the problem of characterising those spaces for which $(C^{\infty}(X), \beta_{\mathcal{S}})$ is separable. Since the Banach space $C^{\infty}(A)$ $(A \in \mathcal{S})$ can only be separable if A is compact (this is a classical result of M. KREIN and S. KREIN and follows from the fact that the Stone-Čech compactification of a non-compact space is never metrisabel – sen GILLMAN and JERISON and $C^{\infty}(A)$ is, at least when X is $\mathcal{S} - \mathcal{S}$ normal, a continuous image of $(C^{\infty}(X), \gamma_{\mathcal{S}})$, it is natural to impose the condition that $\mathcal{S} \subset \mathcal{K}$ i.e. the sets of $\mathcal{S}$ are compact.

Proposition 54 *If $\mathcal{F} \subseteq \mathcal{S} \subseteq \mathcal{K}$ then $(C^{\infty}(X), \beta_{\mathcal{S}})$ is separble if and only if there is a weaker separble, metrisable topology on X.*

Proof. Since we shall use the Stone-Weierstraß theorem, it is convenient to restrict attention to $C^{\infty}_{\mathbf{R}}(x)$. If M is a countable $\beta_{\mathcal{S}}$-dense subset of $C^{\infty}_{\mathbf{R}}(x)$, then the weak topology defined by M satisfies the given conditions.

Sufficiency: let τ be a suitable separable, metrisable topology on X. Then by Urysohn's metristion theorem (WILLARD). (X, τ) can be embedded in a compact, metrisable space Y. For each positive integer n, we can find a finite (diam (U) is the diameter of U). Let ϕ_n be a partition of unity of Y subordinate to $\mathcal{U}_\setminus$ and denote by M the subalgebra of $C^{\infty}_{\mathbf{R}}(X)$. Then M is $\beta_{\mathcal{S}}$-dense by 16 and so $(C^{\infty}_{\mathbf{R}}(x), \beta_{\mathcal{S}})$ is separable.

■

Remark. It follows from SMIRNOV's metrisation theorem (see WILLARD) that if X is locally compact and para-compact and $\mathcal{S}$ possesses a weaker metrisable topology, then X is metrisable. Hence if X is locally compact, paracompact and $(C^{\infty}_{\mathbf{R}}(X), \beta_{\mathcal{S}})$ is separable, then X is metrisable.

Proposition 55 *If X is discrete, then $(C^\infty_{\mathbf{R}}(X), \beta_{\mathcal{S}})$ is separable if and only if card $(X) \leq \text{card}(\mathbf{R})$.*

Proof. The necessity folllows from 54 and the fact that the cardinality of a separable metrisable space is at most card ($\mathbf{R}$). On the otherd and, $\mathbf{R}$ (and hence any subset) has a separable, metrisable topology – the natural topology.

■

Remark. A more intricate argumet shows that the same result holds for metrisable spaces X (see SUMMERS).

In the third section of this chaper we shall consider duality for $C^\infty_(X)$ with strict topologies. However, using the theory of Chapter I, we can already provide some nfromation on this duality, without specifically calculating the dual space – in particular, we cangive sufficient conditons for $\beta_{\mathcal{K}}$ ot be the Mackey topology and we can characterise the relatively weakly compact subsets of $C^\infty_(X)$.

Partitions of unity for $C^\infty(X)$: Let X be a locally compact, paracompact space. Then there exists a partition $\Phi_K : K \in \mathcal{K}\}$ of unity on X so that supp$\Phi_K \subseteq K$. Now $(C^\infty(X), ||\,||, \tau_{\mathcal{K}})$ is the Saks space projective limit of the system $\{C(K) : K \in \mathcal{K}\}$ of Banach spaces and if we define the mappings

$$T_K : x \to (x\Phi_K)^\wedge$$

from $C(K)$ into $C^\infty(X)$ where $(x\phi_K)^\wedge$ denotes the extension of $x\phi_K$ to a function on X obtainded by setting it equal to zero off K, then $\{T_K\}$ is a partition of unity in the sense of ?

Proposition 56 *Let X be locally compact and paracmpact. Then $(C^\infty(X), \beta_K)$ is a Mackey space and has the Banach-Steinhaus property. Also a linear mapping from $C^\infty(X)$ into a separable Fréchet space is $\beta_{\mathcal{K}}$-continuous if and only if its graph is closed.*

In the next results, the phrase "weak topology on $C^\infty(X)$" will be used to denote the weak topology defined by the dual of $(C^\infty(X), \beta_{\mathcal{K}})$.

Proposition 57 *A sequence (x_n) in $C^\infty(X)$ converges weakly to x if and only if $\{x_n\}$ is uniformly bounded and the functions x_n converge pointwise to x.*

Proposition 58 *A bounded subset B of $C^\infty(X)$ is weakly precompact if and only if it is precompact for the topology of pointwise convergence on X. Hence if X is $\mathcal{K}$-complete, then B is relatively weakly compact if and only if it is relatively compact for the topology of pointwise convergence on X.*

PROOF. Using 56, we can reduce 57 and 58 to the case where X is compact. (See, for example GROTHENDIECK, pp. 12 and 209 for this case.)

∎

REMARK. Using results from GROTHENDIECK, one can strengthen 58 as follows: suppose that X is $\mathcal{K}$-complete and has a dense subset which is the union of countably many compact sets. Then the following conditions on a bounded subset B of $C^\infty(X)$ are equivalent:

a) B is relatively countable compact for τ_p (resp. for the weak topology);

b) B is relatively sequentially compact for τ_p (resp. for the weak topology);

c) B is relatively compact for τ_p (resp. for the weak topology).

Here τ_p denotes the topology of pointwise convergence on X.

REMARK. One of the main themes of this Chapter will be that of relating the topological properties of X with the linear (or algebraic) and topological properties of X with the linear (or algebraic) and topological properties of $C^\infty(X)$ with various mixed topologies. We list here some examplex without prooofs:

1. X is hemi-compact (i.e. $\mathcal{K}$ is of countable type) if and only if $B_{\| \|_\infty}$ is $\beta_{\mathcal{K}}$-metrisable;

2. $B_{\| \|_\infty}$ is $\beta_{\mathcal{K}}$-metrisable if and only if X is hemi-compact and each $K \in \mathcal{K}$ is metrisable;

3. if X is locally compact, then $B_{\| \|_\infty}$ is $\beta_{\mathcal{K}}$-separable and metrisable if and only if X is separable and metrisable (alternatively if X is the countable union of compact, metrisable sets);

4. the following conditions are equivalent:

   a) $B_{\| \|_\infty}$ is $\beta_{\mathcal{K}}$-compact (i.e. $(C^\infty(X), \beta_{\mathcal{K}})$ is semi-Montel);
   b) $(C^\infty(X), \beta_{\mathcal{K}})$ is semi-reflexive;
   c) $(C^\infty(X), \beta_{\mathcal{K}})$ is a Schwartz space;
   d) X is discrete.

5. $(C^\infty(X), \beta_{\mathcal{K}})$ is nuclear if and only if X is finite;

6. $\beta_{\mathcal{K}} = \tau_{\mathcal{K}}$ on $C^\infty(X)$ if and only if the union of countable many compact subsets of X is relatively compact. If this is the case, then $(C^\infty(X), \beta_{\mathcal{K}})$ is a (DF)-space.

Remark. A number of results given in this section for $C^\infty(X)$ with the topology $\beta_\mathcal{K}$ (e.g. those of 9 can be extended to $\beta_\mathcal{B}$ with the natural changes. We leave the task of carrying out such extension on the interested reader, metioning only that 56 can be extended to the topology $\beta_\mathcal{B}$ by replacing the assumption of lacal-compactness by local boundedness (obvious definiton) and that SCHMETS and ZAFARANI have studied the topology $\beta_\mathcal{P}$ in [190].

10 Algebras of bounded, continuous functions

In the first part of this section, we work exclusively with the strict topology defined by the family $\mathcal{K}$ of compact subsets of X. To simplify the notation, we denote it by β. First we note that $(C^\infty(X), || \, ||, \tau_\mathcal{K})$ is a pre-Saks algebra (that is, its completion is a Saks algebra).

Proposition 59 *Multiplication is continuous on* $(C^\infty(X), \beta)$.

Proof. We use the representation of β given in 30. If $\varpi \in L^+_\mathcal{K}$ so does $\psi := \sqrt{\varpi}$ and we habe the following inequality

$$p_\varpi(xy) \leq p_\psi(x0p_\psi(y) \quad (x, y \in C^\infty(X)).$$

In general, inversion is not continuous on $C^\infty(X)$ and so $(C^\infty(X), \beta)$ is not a locally multiplicatively convex algebra in sense of MICHAEL.

■

If $t \in X$ then

$$\delta_t : x \to x(t)$$

is an element of the spectrum $M_\gamma(C^\infty(X))$ of $C^\infty(X)$. We habe thus constructed a mapping $\delta : t \to \delta_t$ from X into $M_\gamma(C^\infty(X)$. We call it the **generalised Dirac transformation**. It is injective since $C^\infty(X)$ separates X.

Proposition 60 δ *is a homeomorphism from* X *onto* $M_\gamma(C^\infty(X))$.

Proof. Since the topology on X and $M_\gamma(C^\infty(X))$ are the weak topologies defined by $C^\infty(X)$, it is sufficient to show that δ is surjective. Let f be a β-continuous multiplicative functional on $C^\infty(X)$ and denote by M the kernel of the restriction of f to $C^\infty_\mathbf{R}(X)$. Then there is a $t_0 \in X$ so that $x(t_0) = 0$ for each $x \in M$ (for otherwise M would satisfy the conditions of 16 and so would be β-dense in $C^\infty_\mathbf{R}(X)$ i.e. f would be zero).

■

Note that M separates X. For otherwise there would be points s_1, s_2 in X so that $x(s_1) = x(s_2)$ for $x \in M$. Then M would lie in the kernel of the linear form $x \to x(s_1) - x(s_2)$ and so would habe codimension at least two. Then $M = \{x : x(t_0) = 0\}$ (for both these sets have codimension one) and so $f = \delta_{t_0}$. If X, X_1 are completely regular spaces, $\varpi : X \to X_1$ continuous, then

$$C^\infty(\varpi) : x \to x \circ \varpi$$

is a β-continuous star homomorhism from $C^\infty(X_1)$ into $C^\infty(X)$. In fact, every such homomorhism has this form as the following result shows:

Proposition 61 *If ϕ is β-continuous homomorhism from $C^\infty(X_1)$ into $C^\infty(X)$ then ϕ has the form $C^\infty(\varpi)$ for some continuous mapping ϖ from X into X_1.*

PROOF. If $t \in X$ then $\delta_t \circ \phi$ is a β-continuous multiplicative form on $C^\infty(X_1)$ and so is defined by a unique element of X_1 — we denote this element by $\varpi(t)$. By the construction of this mapping we have

$$\phi(x) = x \circ \varpi$$

for each $x \in C^\infty(X_1)$. Hence for each $x \in C^\infty(X_1)$, $x \circ \varpi \in C^\infty(X)$ and this property characterises continuity for mappings between completely regular space.

∎

Corollar 17 *X and X_1 are homeomorphic if and only if $C^\infty(X)$ and $C^\infty(X_1)$ are isomorphic as pre-Saks algebras.*

We remark that the following version of the Banach-Stone theorem for non-compact spaces can be deduced from 17: if there is an isometry from $C^\infty(X)$ onto $C^\infty(X_1)$ which is also β-bicontinuous, then X and X_1 are homeomorphic.

If $(A, || \, ||, \ )$ is a commutative pre-Saks algebra with unit and if $x \in A$, then the mapping

$$\hat{x} : f \to f(x)$$

from $M_\gamma(A)$ into $\mathbf{C}$ is an element of $C^\infty(M_\gamma(A))$. Thus we have constructed an algebra homomorhism from A into $C^\infty(M_\gamma(A))$. We call it the **generalised Gelfand-Naimark transform**. Note that we canregard $M_\gamma(A)$ as a subspace of the spectrum $M(A)$ of the normed algebra $(A, || \, ||)$. The generalised Gelfand-Naimark transform is then the composition of the Gelfand-Naimark transform for A and the restriction operator from $C(M(A))$ into

$C^\infty(M_\gamma(A))$. In particular, if a is a pre-Saks C^*-algebra, then the generalised Gelfand-Naimark transform is a star-homomorphism (this also follows directly from 61).

Proposition 62 *If $(A, || \, ||, \tau)$ is a commutative Saks C^*-algebra then the generalised Gelfand-Naimark transform is an algebra isomorphism from A onto $C^\infty(M_\gamma(A))$.*

PROOF. We first note that the image of A is $C^\infty(M_\gamma(A))$ is a self-adjoint, separating subalgebra which contains the constants and so is β-dense by the complex version of 16. Now let P be a family of C^*-seminorms on A which define τ (as in ?). For each $p \in P$, we denote by A_p the associated C^*-algebra and by $M(A_p)$ its spectrum. We canregard $M(A_p)$ as a (compact) subset of $M_\gamma(A)$ and we show that $M_\gamma(A) = \bigcup_{p\in P} M(A_p)$. If $f \in M_\gamma(A)$, then, by Duality, there is an increasing sequence (p_n) in P and an $f_n \in A_{p'_n}$ so that $\sum f_n$ is absolutely summable to f. We can also suppose that $||\sum f_n|| < 1+\epsilon$ for an arbitrary positive ϵ. Choose n_0 so that $\sum_{n>n_0} ||f_n|| < \epsilon$. Then $f \in M(A_{p_{n_0}}$ for small enough ϵ. For if $f \notin M(A_{p_{n_0}})$ then there is an $x \in C^\infty(M_\gamma(A))$ so that $||x|| \leq 1$, $x(f) = 1$ and $x = 0$ on $M(A_{p_{n_0}})$. By the β-density of the image of A, there is an $x - 1 \in A$ with $||x_1|| \leq 1+\epsilon$ and $|\hat{x}_1(f)| \leq 1-\epsilon$, $|\hat{x}_1| < \epsilon$ on $M(A_{p_{n_0}})$. Hence, for each $g \in A_{p'_{n_0}}$ with $||g|| < 1+\epsilon$ we have

$$||f-g|| \geq (1+\epsilon)^{-1} ||f(x_1) - g(x_1) \geq \frac{1-\epsilon}{1+\epsilon} - \epsilon$$

and we obtain a contradiction for small ϵ by taking $g = \sum_{n=1}^{n_0} f_n$.

To complete the proof, we let $\mathcal{S}$ be the family of closed subsets of $M_\gamma(A)$ which are containded in some $M(A_p)$ (the p depending on the subset) and, as a temporary notation, $\hat{A}$ be the Saks space projective limit of the systems $\{C(M(A_p))\}_{p\in P}$.

■

Consider the following diagram

file=bild14.eps,height=3cm,width=7cm

where the vertical arrows are the corresponding Gelfand-Naimark transforms and so are isomorphisms. Then the general General transform, being the unique arrwo from A into $\hat{A}$ which preserves commutativity, is an isomorphism an so $\hat{A} = C^\infty(M_\gamma(A))$ and the generalised Gelfand-Naimark transform is surjective.

Note that the inverse of the generalised Gelfand-Naimark transform is β-continuous. However, we cannont, in general, expcet it to be bi-continuous. For example, if $\mathcal{S}$ is a proper subfamily of $\mathcal{K}$ which contains $\mathcal{F}$ and si such that a function $x : X \to ?$ is continuous if and only if its restriction to the sets of $\mathcal{S}$ are continuous, then the general Gelfand-Naimark transform for $(C^\infty(X), || \, ||, \tau_{\mathcal{S}})$ is (up to the obvious identifications) the identity from $(C^\infty(X), || \, ||, \tau_{\mathcal{S}})$ into $(C^\infty(X), || \, ||, \tau_{\mathcal{K}})$ and this is not continuous in general (as an example of such an $\mathcal{S}$ we could take the family consisting of the ranges of convergent sequences and their limit points in a metrisable space).

We now characterise local compactness for X in terms of properties of $C^\infty(X)$. Let $(A, || \, ||, \tau)$ be a commutative Saks algebra and let P be a suitable family of submultiplicative seminorms defining τ. If $p \in P$, put

$$\begin{aligned} I_p &:= \{x \in A : p(x) = 0\} \\ A(I_p) &:= \{y \in A : yI_p = 0\}. \end{aligned}$$

A is **perfect** if $\sum_{p \in P} A(I_p)$ is γ-dense in A. Obviously this property is preserved if we refine the topology τ. As an example, if p is seminorm p_K $(K \in \mathcal{K})$ on $C^\infty(X)$, then

$$A(I_p) = \{x \in C^\infty(X) : x(t) = 0 \text{ for } t \in X \setminus K\}.$$

Hence $A(I_p)$ is $C_c(X)$, the space of functions in $C^\infty(X)$ with compact support.

Proposition 63 *A completely regular space is locally compact if and only if* $C^\infty(X), || \, ||, \tau_{\mathcal{K}})$ *is perfect.*

Proof. In view of the above remarks, this is equivalent to the following statement: X is locally compact if and only if $C_c(X)$ is β-dense in $C^\infty(X)$.

Suppose that X is locally compact. Then $C_c(X)$ separates X and so is β-dense by 16.

Now suppose that $C_c(X)$ is β-dense in $C^\infty(X)$. If $t \in X$, then there is an $x \in C_c(X)$ so that $x(t) > 0$. Then

$$\{s : x(s) > 0\}$$

is relatively compact neighbourhood of t.

Using the generalised Gelfand-Naimark transform, it is easy to see that if A is a perfect, commutative Saks C^*-algebra, then $M_\gamma(A)$ is locally compact. The reserve implication is not true.

Let I be an ideal in $C^\infty(X)$ and write

$$Z(I) = \bigcap_{x \in I} Z(x)$$

where $Z(x) := x^{-1}(0)$ is the zero-set of x. We put

$$I(Z(I)) := \{y^\infty(X) : y = 0 \text{ on } Z(I)\}.$$

Then $I(Z(I))$ is obviously a β-closed ideal and $I \subseteq I(Z(I))$. We shall now show that $I = I(Z(I))$ if and only if I is β-closed. This result is well-known for compact X and we shall use this fact in our proof of the general case.We sketch briefly how it can be proved. Suppose that $x \in I(Z(I))$. For each $\epsilon > 0$ we can find an open neighbourhood U of $Z(I)$ and a function x_ϵ in $c(X)$ so that x_ϵ vanishes on U and $||x - x_\epsilon|| \leq \epsilon$. We shall show that $x_\epsilon \in I$ which will finish the proof. By a compactness argument, there exists $x - 1, \ldots, x_n$ in I so that $U \subset \bigcap_{i=1}^n Z(x_1)$. Then U contains the zero-set of the element $y := \sum_i |x_i|^2$ of I. But then the zero of y and so x_ϵ is a multiple of y.

■

Proposition 64 *Let I be a β-closed ideal of $C^\infty(X)$. Then*

$$I = I(Z(I)).$$

PROOF. I is a norm-closed ideal in $C(\beta X)$ and so there is a closed set K_0 in βX so that $I = \{x \in C(\beta X) : x = 0 \text{ on } K_0\}$.

It is obviously sufficient to show that $K_0 = cl_{\beta X} Z(I)$ (closure in βX) for then if a fuction vanishes on $Z(I)$ its extension to βX vanishes on K_0 and so is in I. If this were not the case, there would be a $t_0 \in K_0 \setminus cl_{\beta X} Z(I)$. Then there is a $y_0 \in C(\beta X)$ with $y_0(t_0) = 1$ and $y = 0$ on a neighbourhood of $cl_{\beta X} Z(I)$.

■

We now show that $y_0 \in I$ which gives a contradiction. To do this, we show that for each $K \in \mathcal{K}$, $\epsilon > 0$, there is a $y_{K,\epsilon}$ in I so that $p_K(y_0 - y_{K,\epsilon}) \leq \epsilon$ and $||y_{K,\epsilon}|| \leq ||y_0||$. Then $(y_{K,\epsilon})$ is a net in I which is $\_ - convergenttoy_0$. To construct $y_{K,\epsilon}$ we proceed as follows: let I_K denote the projection of I in $C(K)$. Then I_K is an ideal in $C(K)$ and so $\bar{I}_K$, its closure in the Banach space $C(K)$, is a closed ideal in $C(K)$. Hence it has the form $\{y \in C(K) : y = 0$ on $Z(I) \cap K\}$. Hence $y|_K \in \bar{I}_K$ and so Tietze's theorem implies the existence of the required $y_{K,\epsilon}$.

The results of this section can be used to give a natural construction of the Stone-Čech compactificaiton and the real-compactification of a completely regular space. We describe this briefly, firstly to display the connection between mixed topologies and the theory of topological extensions and secondly because it will allow us togive a significant generalisation of 60.

If X is a completely regular space, $(C^\infty(X), || \, ||)$ is a Banach algebra. Its spectrum $M(C^\infty(X)))$ is a compact space which we denote by βX. The Dirac transformation can be regarded as a (topological) embedding of X into βX. It has the following universal property: if ϖ is a continuos mapping from X into a comapct space K, then there is a unique continuos extension $\tilde{\varpi}$ of ϖ to a continuos mapping from βX into K. For consider the operator

$$C^\infty(\varpi) : C(K)^\infty(X) = c(\beta X)$$

which is $|| \, || \underline{-} continuos and so (by the closed graph theorem, or more elementarily, by 30)$——— ———- ——— ———$continuos. Hence, by 61,C^\infty(\varpi)$ (regarded as a mapping from $C(K)$ into $C(\beta X)$), has the form $C^\infty(\tilde{\varpi})$ for some $\tilde{\varpi} : \beta X \to K$. $\tilde{\varpi}$ has the required property.

Now we denote by $I(X)$ the set of those functions in $C^\infty(X)$ which habe no zeros in X (i.e. are invertible in the algebra $C^\infty(X)$). Every $x^\infty(X)$ has a unique extension to a function in $C(\beta X)$ which we shall continue to denote by x. Then we put

$$\nu X := \bigcap_{x \in I(X)} C_{\beta X}(x)$$

where $C_{\beta X}(x) = \{s \in \beta X : x(x) \neq 0\}$ is the co-zero set of X in βX.

νX is the **real-compactification** of X and X is **real-compact** if $\nu X = X$. The above rather unfamiliar definition is the natural one from the point of view of strict topologies.

The following equivalent forms are better known:

Proposition 65 *1. $\nu X = \bigcap_{x \in C_{\mathbf{R}}(X)} \tilde{x}^{-1}(\mathbf{R})$ where $\tilde{x}$ denotes the extension of $x \in C_{\mathbf{R}}(X)$ to a function from βX into the 2-point compactification of $\mathbf{R}$.*

2. νX is the completion of X with respect to the $C(X)$-uniformity on X.

Proof. 1. follows from the simple fact that if $x \in I(X)$, then $1/|x| \in C_{\mathbf{R}}(X)$ and the zeros of x in $\underline{X}$ are precisely the points where $(1/|x|)^\sim$ is infinite in value.

For 2. see GILLMAN and JERISON [87], § 15.13.

∎

Corollar 18 *For a completely regular space X, the following are equivalent:*

1. *X is complete for the $C(X)$-uniformity;*
2. *X is real-compact;*
3. *for each $s \in \beta X \setminus X$, there is an $x \in I(X)$ with $x(s) = 0$.*

Now it is clear that the bounded subsets of X are precisely those subsets of X which are precompact in the $C(X)$-uniformity or, by the above result, relatively compact in νX.

This remark makes it natural to generalise the definitions of 9 to include saturated families $\mathcal{S}$ of subsets of νX (for reasons which will be clear later, it is convenient to drop the assumption that the sets be closed). Hence if $\mathcal{S}$ is such a family, we can define the strict topology $\_\mathcal{S}$ on $C^\infty(X)$. We shall always assume that the subsets of $\mathcal{S}$ are relatively compact in νX. The following result is a significant generalisation of 60.

Proposition 66 *The spectrum of the topological algebra $(C^\infty(X), \_\mathcal{S})$ is $\bigcup_{B \in \mathcal{S}} cl_{\nu X}(B)$ (closure in νX).*

PROOF. It is clear that any point in $\bigcup_{B \in \mathcal{S}} cl_{\nu X}(B)$ defines an element in the required spectrum. On the other hand, any point in teh spectrum is defined by a member of νX (apply 60 to the space νX). Hence it is sufficent to show that if $t_0 \notin \bigcup_{B \in \mathcal{S}} cl_{\nu X}(B)$ then δ_{t_0} is not $\_\mathcal{S}$-continuous. But this follows easily from the fact that for each $B \in \mathcal{S}$ there is an $x \in B_{\| \|_\infty}$ so that $x(t_0) = 1$ and $x = 0$ on B.

■

Corollar 19 *The spectrum of $(C^\infty(X), \_\mathcal{S})$ is $\bigcup_{B \in \mathcal{S}} cl_{\nu X}(B)$.*

REMARK. The space of 19 has been introduced by BUCHWALTER [39] who denoted it by X'' (because of a certain format analog with the bidual of a locally convex space) in connectic with the concept of a **μ-space** i.e. a completely regular space in which $\mathcal{B} = \mathcal{K}$ (cf. the concept of semi-Montel locally convex as the limit of the transfinite series $X, X''(X''), \dots$ and X is a μ-space if and only if $X = \mu X$ (or alternatively if $X =$). It is now clear that X is a μ-space if and only if $\_\mathcal{K} = \_\mathcal{B}$ on $C^\infty(X)$.

11 Duality Theory

A classical result of BUCK for the space $C^\infty(X)$ (X locally compact) is that the dual of $(C^\infty(X), \beta)$ is the spce of bounded Radon measures on X. In this section we shall extend this result to completely regular spaces. We shall take this opportunity to discuss various definitions of Radon measures on completely regular spaces.

Definition 14 *A* **preasure** *on X is a member of the (vector space) projective limit of the system*

$$\{\rho_{K_1,K} : M(K_1) \to M(K); K \subseteq K_1, K, K_1 \in \mathcal{K}(\mathcal{X})\}.$$

In other words, a premeasure is a system $\mu = \{\mu_K\}$ of Radon measures which satisfies the compatibility relations $\mu_{K_1}|_K = \mu_K$ ($K \subseteq K_1$). If $\mu = \{\mu_K\}$ is a premeasure on X, $|\mu_K|^*$ denotes the outer measure on K defined by $|\mu_K|$ (see BOURBAKI [25] § VI.1.4).

Thus $\mu_K|^*$ is defined as follows: if $U \subseteq K$ is open, $|\mu_K|^*(U)$ is defined to be $\sup\{\int f d|\mu_K|\}$ where f ranges over the family of positive, continuous functions on K with $f \leq \chi_U$. For general $A \subseteq K$, $|\mu_K|^*(A)$ is defined to be

$$\inf\{|\mu_K|^*(U) : U \text{ open in } K, A \subseteq U\}.$$

Now if C is a subset of X we define $|\mu|^*(C)$ to be $\sup\{|\mu_K|^*(C \cap K)LK \in \mathcal{K}(\mathcal{X})\}$.

A premeasure μ on X is said to be **tight** if for each $\epsilon > 0$ there is a $K \in \mathcal{K}(\mathcal{X})$ so that $|\mu|^*(X \setminus K) < \epsilon$. An equivalent condition is the existence of an increasing sequence $\{K_n\}$ in $\mathcal{K}(\mathcal{X})$ so that $|\mu|^*(X \setminus K_n) \to 0$. We denote by $M_t(X)$ the space of tight measures on X. It is clearly a vector space. If $x^\infty(X)$, $mu \in M_t(X)$, then the limit $\lim_{n\to\infty} \int x|_{K_n} d\mu_{K_n}$ exists and is independent of the particular choise of (K_n). We write $\int x d\mu$ for this limit.

If $K \in \mathcal{K}(\mathcal{X})$ and $\mu \in M(K)$, then μ defines a tight measure on X in a natural way: if $K_1 \in \mathcal{K}(\mathcal{X})$ and $K_1 \subseteq K$ we define μ_{K_1} to be the restriction of μ to K_1. If $K_1 \subset K$ we define μ_{K_1} to be the measure induced on K_1 by μ (for example, as a linear form on $C(K_1)$, μ_{K_1} is the mapping

$$x \to \int x|_{K_1} d\mu).$$

Then $\tilde{\mu} := \{\mu_{K_1}\}$ is a tight measure on X and $|\tilde{\mu}|^*(X \setminus K) = 0$. Hence we can (and do) identify the space of measrues on K with the subspace of $M_t(X)$ consisting of those μ for which $\mu|^*(X \setminus K) = 0$. We denote by $M_0(X)$ the subspace $\bigcup_{K\in\mathcal{K}(\mathcal{K})} M(K)$ — the space of measures with **compact support**. Then $\mu \in M_0(X)$ if and only if there is a $K \in \mathcal{K}(\mathcal{K})$ so that $\mu|^*(X \setminus K) = 0$.

Proposition 67 *The dual of* $(C^\infty(X), \tau_{\mathcal{K}})$ *is naturally isomorphic to* $M_0(X)$ *under the bilinear form*

$$(x, \mu) \to \int x d\mu.$$

PROOF. $(C^\infty(X), \tau_{\mathcal{K}})$ is a dense subspace of the locally convex projective limit of the system

$$\rho_{K_1,K} : C(K_1) \to C(K); K \subseteq K_1, K, K_1 \in \mathcal{K}(\mathcal{X})\}.$$

Now by standard results on the duals of projective imits (see SCHAEFER, Ch. I [167] § IV.4.4.), the dual of the latter space is the union of the spaces $\{M(K)_{K\in\mathcal{K}(\mathcal{X})}$ i.e. $M_0(X)$ under the above identification.

■

Proposition 68 *The dual of* $(C^\infty(X), \beta)$ *is isomorphic to the space* $M_t(X)$ *under the bilinear form*

$$(x, \mu) \to (\int x d\mu.$$

PROOF. Each $\mu \in M_t(X)$ defines a linear form on $C^\infty(X)$ and we show that it is $\tau_{\mathcal{K}}$-continuous on the unit ball of $C^\infty(X)$. If $\epsilon > 0$, choose $K \in \mathcal{K}(\mathcal{X})$ so that $|\mu|^*(X \setminus K) < \epsilon$. Then if

$$\begin{aligned} \left| \int x d\mu \right| &\leq \left| \int_K + \int_{X\setminus K} x d\mu \right| \\ &\leq \epsilon + |\mu|^*(X) \cdot \epsilon \end{aligned}$$

and $|\mu|^*(X) < \infty$ since μ is tight.

Now let f be a β-continuous linear form on $C^\infty(X)$. Then we can express f as a sum $\sum f_n$ where f_n is a continuous linear form on some $C(K_n)$ and $\sum_n ||f_n|| < \infty$. We can regard f_n as a premeasure $\{\mu_K^n\}$ as above. Now $\{\mu_K^n\}_{n\in\mathbf{N}}$ is absolutely summable (in the banach space $M(K)$) and so there is a $\mu_K \in M(K)$ with $\mu_K = \sum \mu_K^n$. It is easy to see that $\mu := \{\mu_K : K \in \mathcal{K}(\mathcal{X})\}$ is a premeasure on X. If we let $K_n^1 := \bigcup_{k=1}^n K_k$ then

$$|\mu|^*(X \setminus K_n^1) \leq \sum_{k>n} ||f_k||$$

and so μ is tight. One can check that $f(x) = \int x d\mu$ for $x^\infty(X)$.

■

12 Alternative definitions of Radon measures

There are several alternative, equivalent definitions for (bounded) Radon measures on a completely regular space and, before continuig, we describe the most important of these. For convenience, we consider only non-negative measures:

A. A **compact-regular Borel measure** μ on X is a -additive finite measure on the Borel algebra of X so that for each Borel set A in X

$$\mu(A) = \sup\{\mu(K) : K \in \mathcal{K}(\mathcal{X}), \mathcal{K} \subseteq \mathcal{A}\}.$$

B. A **Choquet measure** on X is a bounded set funciton $\mu : \mathcal{K}(\mathcal{X}) \to \mathbf{R}_+$ which is increasing, additive (i.e. $\mu(K_1) + \mu(K_2) = \mu(K_1 \cup K_2) + \mu(K_1 \cap K_2)$ for each pair K_1, K_2 of compacta) and continuous on the right (i.e. for $\epsilon > 0$, $K \in \mathcal{K}(\mathcal{X})$ there is neighbourhood V of K os that $\mu(K_1) \leq \mu(K) + \epsilon$ for each $K_1 \in \mathcal{K}(\mathcal{X})$ with $K \subseteq K_1 \subseteq V$).

C. A **tight measure** on X is a bounded, Borel measure which satisfies the tightness condition: for every $\epsilon > 0$, there is a compact set K so that $\mu(X \setminus K) < \epsilon$.

D. A Radon measure μ on βX is **concentrated** on X if $\inf\{\mu(U) : U$ open and $\beta X \setminus X \subseteq U\} = 0$.

Then one can show that the above concepts all coincide in a natural way and corresponds exactly to the non-negative elements of $M_t(X)$. A precise discussion can be found in SCHWARTZ [191].

In teh next Proposition, we can characterise the β-equicontinuous subsets of $M_t(X)$.

Definition 15 *We remark that if $\mu = \{\mu_K\}$ is a tight measure on X then so is the premeasure $\{|\mu_K|\}$. We denote it by $|\mu|$. A subset B of $M_t(X)$ is* **uiformly tight** *if it is bounded (for the norm) in $M_t(X)$ and satisfies the tightness condition:*

> *for every $\epsilon > 0$ there is a $K \in \mathcal{K}(\mathcal{X})$ so that $|\mu|(X \setminus K) < \epsilon$ for each $\mu \in B$.*

Proposition 69 *A subset B of $M_t(X)$ is uniformly tight if and only if it is β-equicontinuous.*

PROOF. We remark firstly that it follows easily from the characterisation of equicontinuous subsets in the dual of a locally convex projective limit of Banach spaces that a subset B_1 of $M_0(X)$ is τ_K-equicontinuous if and only if it is norm bounded and has compact support (i.e. there is a $K \in \mathcal{K}(\mathcal{X})$ so that each $\mu \in B$ vanishes on $X \setminus K$). The result follows then from this fact. ■

Corollar 20 *A uniformly tight subset of $M_t(X)$ is relatively compact for the weak topology defined by $C^\infty(X)$.*

Note that the converse of this result is not always true. In fact, the truth of the converse is equivalent to $(C^\infty(X), \beta)$ being a Mackey space. Hence, the first claim of 56 can be restated as follows:

Proposition 70 *Let X be a locally compact, paracompact space. Then a weakly compact subset of $M_t(X)$ is uniformly tight.*

We now consider properties of the linear operator $C^\infty(\varpi)$ induced by a continuous mapping $\varpi : X \to Y$. We denote by $M_t(\varpi)$ the transposed mapping of $C^\infty(\varpi)$ so that $M_t(\varpi)$ is a norm-bounded linear mapping from $M_t(X)$ into $M_t(Y)$. Note that if we regard a measure $\mu \in M_t(X)$ as a Borel measure then $M_t(\varpi)(\nu)$ is the Borel measure

$$A \to \mu(\varpi^{-1}(A))$$

i.e. it coincides with the measure induced by ϖ in the classical sense.

Proposition 71

1. *$C^\infty(\varpi)$ is a quotient mapping from the Banach space $C^\infty(Y)$ onto a norm-closed subspace of $C^\infty(X)$;*
2. *an element μ in $M_t(Y)$ is in the range of $M_t(\varpi)$ if and only if for each $\epsilon > 0$ there is a $K \in \mathcal{K}(\mathcal{X})$ so that $|\mu|(Y \setminus \varpi(K)) < \epsilon$. $M_t(\varpi)$ is a quotient mapping from $M_t(X)$ onto a norm-closed subspace of $M_t(Y)$;*
3. *$C^\infty(\varpi)$ is an open mapping (for the strict topologies) from $C^\infty(X)$ onto its range if and only if $\varpi(X)$ is closed inY and for each $K \in \mathcal{K}(\varpi(\mathcal{X}))$ there is a $K_1 \in \mathcal{K}(\mathcal{X})$ with $\varpi(K_1) \subseteq K$.*

71 is proved by means of a series of Lemmas. To simplify the notation, we denote the operators $C^\infty(\varpi)$ and $M_t(\varpi)$ by U and V respectively.

Lemma 10 *71 1. holds.*

PROOF. We show that if $y^\infty(X)$ has the form $x \circ \varpi$ for some $x \in C^\infty(Y)$ then there is a $z \in$ with $||z|| = ||y||$ and $y = z \circ \varpi$. But this is the case for z defined as follows:

$$z(t) = x(t) \text{ if } |x(t)| \leq ||y|| x(t)||y||/|y(t)| \text{ otherwise.}$$

■

Lemma 11 *Suppose that X and Y are compact and ϖ is surjektive. Then if $\mu \in M_t(Y)$ there is a $\nu \in M_t(X)$ with $||\mu|| = ||\nu||$ and $V\nu = \mu$.*

PROOF. Since ϖ is surjektive, U is an injection and so an isometry from $C(Y)$ onto a closed subspace A of $C(X)$. We can then regard μ as a continuous linear form on A and the result then follows by taking ν to be a Hahn-Banach extension of this functional to $C(X)$.

■

Lemma 12 *$\mu \in M_t(Y)$ is in the range of V if and only if for each $\epsilon > 0$ there is a compact set K in X so that $|\mu|(Y \setminus \varpi(K)) < \epsilon$. Then there is a $\nu \in M_t(X)$ with $\mu = V\nu$ and $||\nu|| = ||\mu||$.*

PROOF. Necessity: suppose that $\mu = V\nu$ with $\nu \in M_t(X)$. Then for $\epsilon > 0$, there is a $K \in \mathcal{K}(\mathcal{X})$, so that $|\nu|(X \setminus K) < \epsilon$. Then clearly $|\mu|(Y \setminus \varpi(K)) < \epsilon$.

Sufficiency: we can choose an increasing sequence (K_n) of compacta in X so that $|\mu(Y \setminus \varpi(K_n)) < 1/n$. Let

$$A_1 := \varpi(K_1), \quad A_n := \varpi(K_n) \setminus \varpi(K_{n-1}) \qquad (n > 1)$$

and put $\mu_n := \mu|_{A_n}$ (A_n is a Borel set in Y). Then one has $||\mu|| = \sum ||\mu_n||$.

By applying 11 successively to the restrictions of ϖ to K we get a sequence (ν_n) of Radon measure where $\nu_n \in M_t(K_n)$. $||\nu_n|| = ||\mu_n||$ and $V\nu_n = \mu_n$. Then the series $\sum \nu_n$ is absolutely summable in $M_t(X)$ and its sum ν is the required measure. In addition, we have

$$||\nu|| \leq \sum ||\nu_n|| = \sum ||\mu_n|| = ||\mu||.$$

■

Lemma 13 *If $V(M_t(X))$ is weakly closed in $M_t(Y)$, then $\varpi(X)$ is closed in Y.*

PROOF. If $x \in$ which $Ux = 0$, then x vanishes on $\varpi(X)$ and so on $\overline{\varpi(X)}$. hence if $s \in \overline{\varpi(X)}$, then δ_s is in teh point of the kernel of U. But the latter set is $V(M_t(X))$ by the bipolar theorem and so $\delta_n = V\mu$ for some $\mu \in M_t(X)$. Then

$$1 = \delta_s(\{s\}) = V\mu(\{s\}) = \mu(\varpi^{-1}(s))$$

and so $\varpi^{-1}(\{s\})$ is non-empty i.e. $s \in \varpi(X)$.

■

Lemma 14 *If every β-equicontinuous subset in $VM_t(X)$ is the image of a β-equicontinuous set in $M_t(X)$, then each $K \in \mathcal{K}(\varpi(\mathcal{X}))$ is contained in the image of a compact subset of X.*

PROOF. For such a K, let $B := \{\delta_s : s \in K\}$. Then B is clearly uniformly tight and so β-equicontinuous. Hence it is theimage of a β-equicontinuous subset B_1 of $M_t(X)$. Then there is a K_1 in $\mathcal{K}(\mathcal{X})$ so that $|\mu|(X \setminus K_1) < 1/2$ for each $\mu \in B_1$. If $s \in K$ and $\mu \in B_1$ with $V\mu = \delta_s$, then

$$1 = \delta_s(\{s\}) = V\mu(\{s\}) = \mu(\varpi^{-1}(\{s\})$$

and so $\varpi^{-1}(\{s\}) \nsubseteq X \setminus K_1$. Hence $\varpi(K_1) \subseteq K$.

■

To complete the proof of 71, we require the following standard result on locally convex spaces:

Lemma 15 *A continuous linear operator T from a locally convex space E into a locally convex space F is an open mapping into its range if and only if $T(F')$ is (E', E)-closed in E' and each equicontinuous subset of $T'(F')$ is the image of an equicontinuous subset of F'.*

PROOF. See GROTHENDIECK [102].

■

PROOF. of 71 Only 71 3. remains to be proved. The necessity of the given condition follows from 13, 14 and 15.

Sufficiency: first we note that the polar B of $VM_t(X)$ in is the set $\{x \in : x = 0 \text{ on } \varpi(X)\}$. Suppose that $\mu \in B^0$. We show that $\mu \in VM_t(X))^\infty$ is weakly closed. There are compact sets K_n in$X$ so that $|\mu|(X \setminus K_n) \leq 1$. We show that $|\mu|(K_n \setminus \varpi(X)) = 0$ for each n so tht

$$|\mu|(X \setminus (K_n \cap \varpi(X)) \to 0$$

which implies (by 12) that $\mu \in VM_t(X)$. If this were not the case, there would be an n so that

$$\delta := |\mu|(K_n \setminus \varpi(X)) > 0.$$

Choose m so that $m > 2/\delta$. There is a continuous function in $C(K_n)$ with $||x|| = 1$, $x = 0$ on $(K_n \backslash \varpi(X))$ and $\int_{K_n} x d\mu > \delta/2$. We can extend x without increasing the norm to a function x in which vanishes on $\varpi(X)$ (and so is the polar of $VM_t(x)$). Then

$$\int x d\mu > \delta/2 - 1 > 0$$

which gives a contradiction.

■

A similar argument, applied to a uniformly tight set C in $VM_t(X)$, produces a uniformly tight set in $M_t(X)$ whose image is C. Hence the sufficiency follows from 15.

As an application of the theory developed in this section, we give a functional analytic proof of PROHOROV's theorem on the existence of projective limits of measures. We suppose that $\{\varpi''_{\beta\alpha} X_\beta \to X_\alpha, \alpha \leq \beta, \alpha, \beta \in A\}$ is a projective spectrum of completely regular spaces and that X is completely regular space with continuous mappings $\varpi_\alpha : X \to X_\alpha$ so that $\varpi_{\beta\alpha} \circ \varpi_\beta = \varpi_\alpha$ $(\alpha \leq \beta)$ (thus the system $\{\varpi_\alpha\}$ corresponds to a continuous mappings from X into the projective limit of the system $\{X_\alpha\}$). Suppose that $\{\mu_\alpha : \alpha \in A\}$ is a compatible system of bounded Radon measures on $\{X_\alpha\}$ (i.e. $\mu_\alpha \in M_t(X_\alpha)$ and $M_t(\varpi_{\beta\alpha}(\mu_\beta) = \mu_\alpha$ for $\alpha \leq \beta$). We seek necessary and sufficient conditions for the existence of a $\mu \in M_t(X)$ so that $M_t(\varpi_\alpha)(\mu) = \mu_\alpha$ for each α.

Proposition 72 *Such a μ exists if and only if*

1. $\sup\{||\mu_\alpha||_{M_t(X_\alpha)} : \alpha \in A\} < \infty$;
2. *for each $\epsilon > 0$ there exists a $K \in \mathcal{K}(\mathcal{X})$ so that $|\mu_\alpha|(X_\alpha \setminus \varpi_\alpha(K)) < \epsilon$ for each α.*

Proof. The necessity of condition 1. is trivial and that of 2. follows from 71 2.

Now suppose that 1. and 2. are satisfied. If $n \in \mathbf{N}$, choose $K_n \in \mathcal{K}(\mathcal{X})$ so that $|\mu_\alpha|(X \setminus \varpi_\alpha(K_n)) < 1/n$. Let

$$B := \{\mu \in M_t(X) : ||\mu|| \leq \sup ||\mu_\alpha|| \text{ and } |\mu(X \setminus K_n) < 1/n\}.$$

Then B is weakly compact in $M_t(X)$. Let

$$B_\alpha := \{\mu \in B : M_t(\varpi_\alpha)(\mu) = \mu_\alpha\}.$$

Then B_α is weakly compact and non-empty by 71 2. Hence $\cap B_\alpha \neq \varpi$ by the finite intersection property.

■

Using 67 and the ideas of 66 and 19, we can give a characterisation of the dual of $(C^\infty(X), \beta_{\mathcal{B}})$ analogous to that of 12 C. for $M_t(X)$. We denote this dual by $M_{\mathcal{B}}(X)$ (so that $M_t(X) \subseteq M_{\mathcal{B}}(X)$).

Proposition 73 *The space $M_{\mathcal{B}}(X)$ can be naturally identified with the space of Radon measure μ on βX which satisfy the following condition: for each $\epsilon > 0$ there is a $B \in \mathcal{B}$ so that $|\mu|(\beta X \setminus \bar{B}) < \epsilon$ ($\bar{B}$ denotes the closure of B in β).*

PROOF. By applying 67 $(C^\infty(\nu X), \beta)$ we see that a $\beta_{\mathcal{B}}$-continuous linear form, which can be regarded as a $\beta_{\mathcal{K}}$-continuous linear form on $C^\infty(\nu X)$, is defined by a Radon measure μ on $|\mu|(\beta X \setminus K) < \epsilon$. Since μ is $\beta_{\mathcal{B}}$-continuous, one can show as in proof of 67 that one can even take K to be of the form $\bar{B}(B \in \mathcal{B}(\mathcal{X}))$.

■

13 Representationof operators on $C^\infty(X)$

In this section, we consider Riesz-type representation theorems for operators from $C^\infty(X)$ into a Saks space. We use the projective limet representation of Saks spaces with compact unit ball to give a simple proof of the result for operators with range in such spaces and deduce some classical results on operators with values in a Banach space F. To this end, we introduce the space $M_t(X; F)$ of tight F-valued measures i.e. the bounded, -additive measures on the Borel sets of X with values in F which are inner regular with respect to compact sets. Then integration defines a β-continuous linear operator $T_\mu : x \to \int x d\mu$ from $C^\infty(X)$ into F. $\mu \to T_\mu$ is an isometry from $M_t(X, F)$ into $(C^\infty(X); F)$ and thus allows us to regard the former as a Saks space by using the auxiliary topology of pointwise convergence on $C^\infty(X)$. In general, the mapping $\mu \to T_\mu$ is not onto as the example of the identity on $C([0, 1])$ shows. However, it is onto if F is finite dimensional — this is a trivial generalistion of 67. In this case we have a natural isometric isomorphism $M_t(X; F) \cong (C^\infty(X), F)$ which is functorial, being given by integration. This identification is also an isomorphism for the Saks space structures on both sides.We note that if E is a Saks space, we can define $M_t(X; F)$ in an analogous manner and provide it with a naturall Saks space structure. Of course, we define the boundedness with respect tot the norm and the regularity with respect to the auxiliary topology. We remark that if the complete Saks space E has a representation as a rojective limit $S - \lim_{\substack{\longleftarrow \\ \alpha}} E_\alpha$ of a spectrum of Banach spaces, then $M_t(X; E) \cong S - \lim_{\substack{\longleftarrow \\ \alpha}} M_t(X; E_\alpha)$.

Proposition 74 *Let $(E, || \, ||, \ )$ be a Saks space with $B_{|| \, ||}$ τ-compact, X a completely regular space. Then if $T : C^\infty(X) \to E$ is a $\beta - \gamma$ continuous linear operator, there exists a Radon measure $\mu : (X) \to E$ representing T i.e. T is the operator*

$$T_\mu : x \to \int x d\mu.$$

Conversely, every Radon measure μ defines a $\beta - \gamma$ continuous linear operator T_μ in the above manner. Hence integration establishes a Saks space isomorphism

$$M_t(X; E) \cong (C^\infty(X), E).$$

Proof. We put $G : E;_\gamma$ and calculate:

$$\begin{aligned} (C^\infty(X), E) &\cong (C^\infty(X), S - \varprojlim_{F \in \mathcal{F}(\mathcal{G})} F') \\ &\cong S - \varprojlim_F (C^b(X), F') \\ &\cong S - \varprojlim_F M_t(X, F') \\ &\cong M_t(X, S - \varprojlim_F F') \\ &\cong M_t(X, E). \end{aligned}$$

■

Remark. The formal manipulations with projective limits used in this proof are justified by the fact that the isomorphisms at each step are implemented by intergration.

A less formal demonstration of the above result goes as follows: since T maps bounded sets in $C^\infty(X)$ into relatively compact subsets of (E, γ), then

$$T'' : C^\infty(X)'' \to E''$$

actually takes its values in E.

Noting that if A is a Borel set in X then its characterist function χ_A defines, by intergration, an element of $C^\infty(X)'' = (M(X), || \, ||)'$ we may define

$$\mu(A) := T''(\chi_A)$$

which is an E-valued measure.

By the continuity of T'' (with respect to the norm in $C^\infty(X)''$ we can deduce that

$$T''(X) = \int x d\mu$$

for every bounded, Borel-measurable function on X and so in particular, for $x^\infty(X)$. The converse fact is easy.

Corollar 21 *If T and X are as above, then the following are equivalent:*

1. *T is $\beta - || \, ||$ continuous;*

2. *T is compact i.e. takes some β-neigbourhood of zero to a relatively compact set in E;*

3. *the semi-variation $||\mu||$ of μ (with respect to the norm E) is tight i.e. for each $\epsilon > 0$ there exists a $K \in \mathcal{K}(\mathcal{S})$ so that for each $A \in (X)$ with $A \subseteq X \setminus K$, $||\mu||(A) < \epsilon$.*

Proof. The equivalence of 1. and 2. follows from Proposition ? (which is trivial in this case since $B_{|| \, ||}$ is compact). The equivalence of 2. and 3. is clear.

∎

Using the above result, we can now easily obtain a result for general operators with values in a locally convex space.

Proposition 75 *Let E be a locally convex space, X a completely regular space. Then any continuous, linear operator $T : C^\infty(X) \to E$ may be represented by integration with respect to a Radonmeasure μ from (X) into $(E'', (E'', E'))$. In fact, μ takes its values in the (E'', E')-closure of $T(B(C^\infty(X)))$.*

If T maps the unit ball of $C^\infty(X)$ into a relatively weakly compact subset of E, then μ takes its values in E (actually in $\overline{T(B, C^\infty(X))}$ and is a Radon measure with respet tot the original topology in E.

Proof. For the first assertion, let B be the (E'', E')-closure of $T(B(C^\infty(X)))$ in E'' and let F be the Saks space spanned by B in E'' with $|| \, ||_B$ as norm and (E'', E') as auxiliary topology. Then this is a Saks space with compact unit ball and the result follows immediately from 74.

In the second case, thake $B := \overline{T(B(C^\infty(X)))}$, the closure now being taken in E, and define the Saks space F to be $(E_B, || \, ||_B, (E, E'))$. Then T is represented by an F-valued Radon measure μ (i.e. Radon with respect to (E, E')).

∎

The fact that μ is Radon with respect to the topology of E follows from the following simple Lemma.

Lemma 16 *Let E be a locally convex space and suppose that*

$$\mu : (X) \to E$$

is a finitely additive measure so that for each f in E', $f \circ \mu$ is a Radon measure. Then μ is also Radon.

We have shown that weakly compact operators on $C^\infty(X)$ have representations as integrals. An example of an operator which does not have such a representation is the identity operator on c_0. Here the representing measure is the measure

$$A \to \chi_A$$

on the power set of **N**. In a certain sense this is **the** typical example as the following generalistion of a result of Pelczyński shows. In the proof we use Grothendiek's characterisation of weakly compact subset of $M_t(X)$ and Rosenthal's Lemma on sequences of measures.

Proposition 76 *Let (E, τ) be a quasi-compelete locally convex space $T : C^\infty(X) \to E$ a β-continuous linear operator. If T does not map the unit balls of $C^\infty(X)$ into a relatively weakly compact subset of E, there is a sequence (x_n) of funcitons $C^\infty(X)$ with mutually disjoint supports so that if j is the mapping $(l_n) \to \sum l_n x_n$ from c_n into $C^\infty(X)$ and A denotes the β-closed span of $\{x_n\}$ in $C^\infty(X)$, then in the following diagram*

file=bild15.eps,height=3cm,width=7cm

j and $T|_A$ are isomorphisms. More informally, T fixes a subspace of $(C^\infty(X), \beta)$ which is isomorphic to c_0.

Consequently, if E fails to contain a copy of c_0 then every continuous linear operator $T : C^\infty(X) \to E$ takes the unit ball into a relatively weakly compact subset of E.
PROOF. If T fails to satisfy the given condition, then $T' : E' \to M_t(X)$ takes some equicontinuous set H in E' to a subset of $M_t(X)$ which is bounded but not relatively $(M(X), M(X)')$-compact. Then, by the above mentioned characterisation of weakly compact sets in $M_t(X)$ there exists a sequence (f_n) in H, a sequence (U_n) of disjoint open sets in S and an $\epsilon > 0$ so that $|T'(f_n)(U_n)| > \epsilon(n \in \mathbf{N})$ i.e. $|f_n \circ \mu(U_n)| > \epsilon$ (where μ represets T). By ROSENTHAL's Lemma we may suppose that

$$|f_n \circ \mu|(\cup_{m \neq n} U_m) < \epsilon/2 \quad (n \in \mathbf{N}).$$

Now choose a sequence (x_n) in $C^\infty(S)$ so that $|x_n| \leq \chi U_n$ and

$$|f_n \circ T(x_n)| = \left| \int_S x_n d(f_n \circ \mu) \right| > \epsilon,$$

which is possible since $f_n \circ \mu$ is a Radon-measure. Then j, as defined in teh statement of the theorem, is clearly a well-defined, continuous injection. We claim that

$$||T \circ j((l_n))||_H \geq \epsilon/2 ||(l_n)||_{c_0}$$

for each $(l_n) \in c_0$, where $|| \, ||_H$ denotes the seminorm of unit form convergence on the equicontinuous set H. Indeed for any $(l_n) \in c_0$ and any $k \in \mathbf{N}$,

$$\begin{aligned} ||T \circ j((l_n))||_H &\geq |T \circ j((l_n)), f_k| = \left| \int_S \sum_{n \in \mathbf{N}} l_n x_n d(f_k \circ \mu) \right| \\ &\geq \left| \int_S l_k x_k d(f_k \circ \mu) \right| - ||(l_n)||_{c_0} |f_k \circ \mu| (\cup_{l \neq k} U_l) \\ &\geq |l_k| \epsilon - ||(l_n)||_{c_0} \cdot \epsilon/2. \end{aligned}$$

Taking the supremum over k on the right-hand side we get the required estimate.

■

This shows that $(T \circ j)^{-1}$ is well-defined and continuous on $T(j(c_0))$, from which it follows that j as an operator from c_0 to $j(c_0)$ and T, as an operator from $j(c_0)$ to $T(j(c_0))$ are isomorphisms (by the following trivial Lemma).

Lemma 17 *Let X, Y, Z be topological spaces, $f : X \to Y$ and $g : Y \to Z$ continuous, surjective mappings such that $g \circ f$ is an isomorphism. Then f and g are also isomorphisms.*

Proof. The injecticity of $g \circ f$ imlies that of g and f that f^{-1} and g^{-1} are well-defined. But as $g^{-1} = f \circ (g \circ f)$ and $f^{-1} = (g \circ f)^{-1} \circ g$ it is clear that f and g are continuous.

■