An explicit class of min–max polynomials on the ball and on the sphere

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1. Introduction

Let $N, d \in \mathbb{N}_0, d \geq 2$, and let $\Pi^d_N$ denote the space of polynomials in $d$ variables of total degree less than or equal to $N$ with real coefficients, i.e.,
\[
\Pi_N^d := \left\{ P : P(x) = \sum_{|\ell| \leq N} c_\ell x^\ell, c_\ell \in \mathbb{R} \right\},
\]

where \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), \( \ell = (l_1, \ldots, l_d) \in \mathbb{N}_0^d \) and \( x^\ell = x_1^{l_1} \cdots x_d^{l_d} \) is the multivariate monomial of total degree \(|\ell| := l_1 + \cdots + l_d\). Furthermore we use the standard notation \( \|x\| = \sqrt{x_1^2 + \cdots + x_d^2} \) for the Euclidean norm of \( x \in \mathbb{R}^d \), \( \langle a, x \rangle = a_1 x_1 + \cdots + a_d x_d \) for the scalar product of \( a, x \in \mathbb{R}^d \), \( B^d := \{ x \in \mathbb{R}^d : \|x\| \leq 1 \} \) for the \( d \)-dimensional unit ball and \( S^{d-1} := \{ x \in \mathbb{R}^d : \|x\| = 1 \} \) for its sphere. In addition we use the notation \( x = (x', x_d) \) for \( x \in \mathbb{R}^d \), with \( x' = (x_1, \ldots, x_{d-1}) \).

For a given homogeneous polynomial of degree \( N \),
\[
\mathcal{P}(x) := \sum_{|\ell| = N} a_\ell x^\ell,
\]

with \( a_\ell \in \mathbb{R} \) fixed, we look for a solution \( p^* \) of the following problem:
\[
\|(\mathcal{P} - p^*)w\|_K = \min_{p \in \Pi_N^d} \|(\mathcal{P} - p)w\|_K,
\]

where \( K \) denotes the region \( B^d, S^{d-1}, w \) is a weight function on \( K \) and
\[
\|(\mathcal{P} - p^*)w\|_K := \max_{x \in K} |(\mathcal{P}(x) - p^*(x))w(x)|.
\]

We call \( \mathcal{P} - p^* \) a min–max polynomial on \( K \) with respect to the weight function \( w \), or simply a min–max polynomial on \( K \) when \( w(x) = 1 \), and \( \|(\mathcal{P} - p^*)w\|_K \) the minimum deviation on \( K \).

While for the unit disc, i.e., for \( d = 2 \), for a wide class of homogeneous polynomials of degree \( N \), min–max polynomials are known for every \( N \in \mathbb{N} \) (see [6,8]), for the ball and the sphere in dimension \( d, d \geq 3 \), only corresponding modified Chebyshev polynomials are known to be min–max polynomials for every \( N \in \mathbb{N} \), as \( T_{2N}(\|x\|) \) and \( T_N(\langle a, x \rangle) \), where \( a = (a_1, \ldots, a_d) \in \mathbb{R}^d, \|a\| = 1 \), belonging to the class of radial and ridge polynomials, respectively. As usual, \( T_k \) denotes the Chebyshev polynomial of the first kind, defined by \( T_k(y) := \cos k \arccos y, k \in \mathbb{N}, y \in [-1,1] \). Otherwise min–max polynomials are known only for some special polynomials of fixed small degree, such as \( x_1^2 x_2^2 \cdots x_d^2 \) in dimension \( d = 3, 4, 5 \) (see [13]), \( x_1^2 x_2 x_3^4 \) in dimension \( d = 3 \) (see [13]), \( x_1 x_2 \cdots x_d, x_1^2 x_2 \cdots x_d, x_1^4 + \cdots + x_d^4 \) in dimension \( d \geq 3 \) (see [12, Example 2, p. 264] and [3, Corollary 1], [2, Theorem 2.4], and [3, Corollary 2] respectively).

2. Main result

Notation. We denote by \( q_m(s; w) = q_m(s; w(s)) := s^m + \cdots, m \in \mathbb{N} \), the monic min–max polynomial on \([0,1]\) with respect to the weight function \( w \), i.e.,
\[
\min_{a_0 \in \mathbb{R}} \max_{s \in [0,1]} |(s^m + a_{m-1}s^{m-1} + \cdots + a_0)w(s)| = \max_{s \in [0,1]} |q_m(s; w)w(s)|
\]

and let
\[
E_{m-1}(s^m; w(s)) := \max_{s \in [0,1]} |q_m(s; w)w(s)|
\]

denote the minimum deviation. For \( m = 0 \), let \( E_{m-1}(s^m; w(s)) := 1 \).

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In the following the product of so-called real homogeneous harmonic polynomials in two variables of degree \( n \), \( P_n(x_1, x_2) \), \( n \in \mathbb{N} \), that is, polynomials of the form

\[
P_n(x_1, x_2) = \alpha \text{Re}\{(x_1 + ix_2)^n\} + \beta \text{Im}\{(x_1 + ix_2)^n\},
\]

where \( \alpha, \beta \in \mathbb{R} \), will play an important role. The basis polynomials for the first few degrees are: \( 1; x_1, x_2; x_1^2 - x_2^2, x_1x_2; x_1(x_1^2 - 3x_2^2), x_2(3x_1^2 - x_2^2); (x_1^2 - x_2^2)^2 - 4x_1^2x_2^2, x_1x_2(x_1^2 - x_2^2); x_1(x_1^4 - 10x_1^2x_2^2 + 5x_2^4), x_2(5x_1^4 - 10x_1^2x_2^2 + x_2^4); \ldots \). For the definition of the homogeneous harmonic polynomials and their importance for Fourier analysis in higher dimensions, see e.g. [4, Chapter 9].

**Theorem 2.1.** Let \( d, m \in \mathbb{N}_0, d \geq 3 \) odd, \( n = (n_1, n_2, \ldots, n_{(d-1)/2}) \in \mathbb{N}^{(d-1)/2} \) and let \( P_{n_1}(x_1, x_2), P_{n_2}(x_3, x_4), \ldots, P_{n_{(d-1)/2}}(x_{d-2}, x_{d-1}) \) be homogeneous harmonic polynomials given by (5). Furthermore, let

\[
M_n := \sqrt{n_1^{n_1}n_2^{n_2} \cdots n_{(d-1)/2}^{n_{(d-1)/2}}} |n|^{n/2}.
\]

(a) The polynomial

\[
\prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k})q_m(x_d^2; (1 - s)^{n/2}) = \prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k})x_d^{2m} + e(x),
\]

where \( e(x) \in \Pi_{|n|+2m-1}^d \) is a min–max polynomial on \( B^d \) and \( S^{d-1} \). The minimum deviation is

\[
\prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; (1 - s)^{n/2}).
\]

(b) The polynomial

\[
\prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k})x_d q_m(x_d^2; s^{1/2}(1 - s)^{n/2}) = \prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k})x_d^{2m+1} + e(x),
\]

where \( e(x) \in \Pi_{|n|+2m}^d \) is a min–max polynomial on \( B^d \) and \( S^{d-1} \). The minimum deviation is

\[
\prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; s^{1/2}(1 - s)^{n/2}).
\]

**Remark 2.2.** We note here that if the polynomial \( P(x) \) defined by (1) depends only on \( k < d \) variables \( x_1, \ldots, x_k \), then

\[
\inf_{p \in \Pi_{N-1}^d} \|P - p\|_{B^d} = \inf_{q \in \Pi_{N-1}^{d-k}} \|P - q\|_{B^k},
\]

and similarly for \( S^{d-1} \) and \( S^{k-1} \). Furthermore, if \( P(x_1, \ldots, x_d) = p^*(x_1, \ldots, x_d) \) is a min–max polynomial on \( B^d \) (respectively \( S^{d-1} \)), then \( P(x_1, \ldots, x_d) = p^*(x_1, \ldots, x_k, 0, \ldots, 0) \).
The polynomial $P(x_1, \ldots, x_d) = q^n(x_1, \ldots, x_k)$ is a min–max polynomial on $B^k$ (respectively $S^k$), and conversely, if $P(x_1, \ldots, x_d) - q^n(x_1, \ldots, x_k)$ is a min–max polynomial on $B^k$ (respectively $S^k$), then it is also a min–max polynomial on $B^d$ (respectively $S^{d-1}$). The detailed proof of this fact can be found in [14, Proposition 4.1]; see also [2, p. 24].

**Remark 2.3.** As a consequence of Theorem 2.1, putting $m = 0$, and Remark 2.2, we obtain the interesting fact that any product $\prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k})$ of harmonic homogeneous polynomials is a min–max polynomial on $B^{d-1}$ and $S^{d-2}$, $d \geq 3$ odd, and the minimum deviation is $\prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n$.

**Theorem 2.4.** Let $d, m \in \mathbb{N}_0$, $d \geq 4$ even, $n = (n_1, n_2, \ldots, n_{(d-2)/2}) \in \mathbb{N}^{(d-2)/2}$ and let $P_{n_1}(x_1, x_2), P_{n_2}(x_3, x_4), \ldots, P_{n_{(d-2)/2}}(x_{d-3}, x_{d-2})$ be harmonic homogeneous polynomials given by (5). Furthermore, let

$$M_n := \sqrt{\frac{n_1 n_2 \cdots n_{(d-2)/2}}{(n + 1)^{(d-2)/2}}}.$$  

(a) The polynomial

$$\prod_{k=1}^{(d-2)/2} P_{n_k}(x_{2k-1}, x_{2k}) x_{d-1} q_m(x_d^2; (1 - s)^{(|n|+1)/2})$$

$$= \prod_{k=1}^{(d-2)/2} P_{n_k}(x_{2k-1}, x_{2k}) x_{d-1} x_d^{2m} + e(x),$$

where $e(x) \in \Pi_{|n|+2m}^d$, is a min–max polynomial on $B^d$ and $S^{d-1}$. The minimum deviation is

$$\prod_{k=1}^{(d-2)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; (1 - s)^{(|n|+1)/2}).$$

(b) The polynomial

$$\prod_{k=1}^{(d-2)/2} P_{n_k}(x_{2k-1}, x_{2k}) x_{d-1} x_d q_m(x_d^2; s^{1/2} (1 - s)^{(|n|+1)/2})$$

$$= \prod_{k=1}^{(d-2)/2} P_{n_k}(x_{2k-1}, x_{2k}) x_{d-1} x_d^{2m+1} + e(x),$$

where $e(x) \in \Pi_{|n|+2m+1}^d$, is a min–max polynomial on $B^d$ and $S^{d-1}$. The minimum deviation is

$$\prod_{k=1}^{(d-2)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; s^{1/2} (1 - s)^{(|n|+1)/2}).$$

**Remark 2.5.** As a consequence of Theorems 2.1 and 2.4 we obtain min–max polynomials for the monomials $x_1 \cdots x_{d-1} x_d^m$ for every $m \in \mathbb{N}$. The very special cases $m = 0, m = 1$, and $m = 2$ have been previously studied by completely different methods in [3, Corollary 1], [12, Example 2, p. 264], and [2, Theorem 2.4], respectively.

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To state our next result, let us recall (see for example [10, p. 3]) that

\[ T_n(y) = \sum_{k=0}^{[n/2]} t_{n-2k} y^{n-2k}, \]

for all \( n \in \mathbb{N}, y \in \mathbb{R}, \) where

\[ t_{n-2k} := (-1)^k \frac{[n/2]}{2^{j}} \binom{n}{2j} \binom{j}{k}, \quad k = 0, \ldots, [n/2]. \]

**Theorem 2.6.** Let \( d, n \in \mathbb{N}, d \geq 3, m \in \mathbb{N}_0, \) and let \( a' = (a_1, \ldots, a_{d-1}) \in \mathbb{R}^{d-1} \) such that \( \|a'\| = 1. \)

(a) The polynomial

\[ \|x'\|^{n} T_n \left( \frac{\langle a', x' \rangle}{\|x'\|} \right) q_m \left( \frac{x'^2}{d}; (1-s)^{n/2} \right) = \left( \sum_{k=0}^{[n/2]} t_{n-2k} \langle a', x' \rangle^{n-2k} \|x'\|^{2k} \right) x'^{2m+1} + e(x), \]

where \( e(x) \in \Pi_{n+2m-1}^{d} \), is a min–max polynomial on \( B_d \) and \( S^{d-1}. \) The minimum deviation is \( E_{m-1}(s^m; (1-s)^{n/2}). \)

(b) The polynomial

\[ \|x'\|^{n} T_n \left( \frac{\langle a', x' \rangle}{\|x'\|} \right) x_d q_m \left( \frac{x'^2}{d}; s^{1/2} (1-s)^{n/2} \right) \]

\[ = \left( \sum_{k=0}^{[n/2]} t_{n-2k} \langle a', x' \rangle^{n-2k} \|x'\|^{k} \right) x'^{2m+1} + e(x), \]

where \( e(x) \in \Pi_{n+2m}^{d} \), is a min–max polynomial on \( B_d \) and \( S^{d-1}. \) The minimum deviation is \( E_{m-1}(s^m; s^{1/2} (1-s)^{n/2}). \)

**Remark 2.7.** We recall that Bernstein [5, p. 225] has shown that for \( \rho_1, \rho_2 \geq 0, \)

\[ E_{m-1}(s^m; s^{\rho_1} (1-s)^{\rho_2}) \sim \frac{1}{2^{2m-1+2\rho_1+2\rho_2}}, \]

where, as usual, \( a_n \sim b_n \) if \( 1 - \varepsilon_n \leq \frac{a_n}{b_n} \leq 1 + \varepsilon_n \) with \( \varepsilon_n \to 0, \) and thus the minimum deviation in Theorems 2.1, 2.4 and 2.6 can be given asymptotically explicitly. The min–max polynomials \( q_m(s; s^{\rho_1} (1-s)^{\rho_2}) \) are known so far for the following few cases only: \((\rho_1, \rho_2) \in \{(0, 0), (1/2, 1/2), (1/2, 0), (0, 1/2), (0, 1), (1, 0)\}.\) As regards \((\rho_1, \rho_2) = (0, 1),\) see Corollary 2.9 below.

By projection (see Proposition 3.1), the min–max polynomials on \( S^{d-1} \) from Theorem 2.1 give min–max polynomials on \( B^{d-1}. \)

**Corollary 2.8.** Let the assumptions of Theorem 2.1 be satisfied. Then:

(a) The polynomial

\[ \prod_{k=1}^{(d-1)/2} P_{n_k}(x_{2k-1}, x_{2k}) q_m \left( 1 - \|x'\|^2; (1-s)^{[n]/2} \right) \]
The polynomial and Proposition 3.1. We even obtain a new class of min–max polynomials on $B^{d-1}$. The minimum deviation is given by (8).

(b) The polynomial
\[
(-1)^m \prod_{k=1}^{(d-1)/2} P_n(x_{2k-1}, x_{2k}) \|x\|^2m + e(x'),
\]
where $e(x') \in \Pi_{|n|+2m-1}^{d-1}$, is a min–max polynomial on $B^{d-1}$. The minimum deviation is given by (10).

Correspondingly the min–max polynomials on $S^{d-1}$ from Theorems 2.4 and 2.6 can be carried over, with the help of Proposition 3.1, into min–max polynomials on $B^{d-1}$.

For dimension $d = 3$ the min–max polynomial $x_1 x_2 x_3^{2m} + \cdots$, or more generally $(\alpha x_1^2 - x_2^2) + \beta x_1 x_2 x_3^{2m} + \cdots$, can even be written down explicitly. Indeed, let $\tilde{T}_m(y) := \frac{1}{2m-1} T_m(y)$, $m \in \mathbb{N}$, be the monic Chebyshev polynomial on $[-1, 1]$. By shifting the largest zero of $\tilde{T}_m(y)$ into the point 1, we obtain the monic polynomial which deviates least from zero on $[-1, 1]$ among all monic polynomials of degree $2m$ which have simple zeros at the points $-1$ and $1$, as can be easily shown with the help of the Chebyshev Alternation Theorem. For a similar use of this trick based on a change of variable and application of Chebyshev’s Alternation Theorem, see [1, p. 280–289, 336]. Thus we obtain:

**Corollary 2.9.** Let $\alpha, \beta \in \mathbb{R}$, and put $k_m = \cos \frac{\pi}{2m}$ and $\tilde{T}_m(y) := \frac{1}{(k_m)^m} \tilde{T}_m(k_m y)$, for $m \in \mathbb{N}$. Then for every $m \in \mathbb{N}_0$,
\[
(\alpha x_1^2 - x_2^2) + \beta x_1 x_2 \frac{\tilde{T}_{2m+2}(x_3)}{x_3} = (\alpha x_1^2 - x_2^2) + \beta x_1 x_2 x_3^{2m} + e(x_1, x_2, x_3),
\]
where $e(x_1, x_2, x_3) \in \Pi_{2m+1}^3$ is a min–max polynomial on $B^3$ and $S^2$.

The very special case $\alpha = 0, m = 1$ can be found in Reimer [9, p. 178, 336].

Finally, we mention that as a special case of Corollary 2.8 we even obtain a new class of min–max polynomials on the disc $D := B^2$.

**Corollary 2.10.** Let $n \in \mathbb{N}, m \in \mathbb{N}_0$, and let $P_n(x_1, x_2)$ be a real homogeneous harmonic polynomial of degree $n$ given by (5).

(a) The polynomial
\[
q_m(1 - x_1^2 - x_2^2; (1 - s)^{n/2}) P_n(x_1, x_2) = (-1)^m (x_1^2 + x_2^2)^m P_n(x_1, x_2) + e(x_1, x_2),
\]
where $e(x_1, x_2) \in \Pi_{2m+n-1}^2$ is a min–max polynomial on $D$. The minimum deviation is $\sqrt{\alpha^2 + \beta^2 E_{m-1}} (s^m; (1 - s)^{n/2})$.

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(b) The polynomial
\[ q_m(1 - x_1^2 - x_2^2; s^{1/2}(1 - s)^{n/2}) P_n(x_1, x_2) \]
\[ = (-1)^m (x_1^2 + x_2^2)^m P_n(x_1, x_2) + e(x_1, x_2), \tag{19} \]
where \( e(x_1, x_2) \in \mathbb{P}_{2m+n-1}^2 \), is a min–max polynomial on \( \mathcal{D} \) with respect to the weight function \( \sqrt{1 - x_1^2 - x_2^2} \). The minimum deviation is \( \sqrt{\alpha^2 + \beta^2 E_{m-1}(s^m; s^{1/2}(1 - s)^{n/2})} \).

**Corollary 2.10(a)** could be also derived with the help of Theorem 4.1 from [6].

### 3. Proofs

The proofs of our main results are based on the characterization of min–max polynomials in terms of the notion of extremal signature due to Rivlin and Shapiro [11, Theorem 2]. We give below the definition of the extremal signature restricted to the setting of our approximation problem.

A real-valued function \( \sigma \) on \( K \) has finite support \( S = \{x^{(1)}, \ldots, x^{(r)}\} \), where \( x^{(k)}, k = 1, \ldots, r \), are distinct points of \( K \), if \( \sigma(x) = 0 \) on \( K \setminus S \) and if \( \sigma(x^{(k)}) \neq 0, k = 1, \ldots, r \). A signature \( \sigma \) is a function with finite support \( S \) whose values at the points \( x^{(k)} \in S \) are \( +1 \) or \( -1 \). A signature \( \sigma \) with finite support \( S = \{x^{(1)}, \ldots, x^{(r)}\} \) is an extremal signature with respect to \( \mathbb{P}_N^d \), if there exist positive real numbers \( \lambda_k, k = 1, \ldots, r \), such that
\[ \sum_{k=1}^r \lambda_k \sigma(x^{(k)}) Q(x^{(k)}) = 0, \quad \text{for all } Q \in \mathbb{P}_N^d. \]

For some examples of extremal signatures in several dimensions see [7,12].

First we prove a connection between the min–max polynomials on \( S^{d-1} \) when the homogeneous polynomial \( \mathcal{P}(x) \) given by (1) is even in \( x_d \), that is,
\[ \mathcal{P}(x) := \sum_{\|x\|_N = 0} a_q x^1, \tag{20} \]
or odd in \( x_d \), i.e.,
\[ \mathcal{P}(x) := \sum_{\|x\|_{N+1} = 0} a_q x^1, \tag{21} \]
and the min–max polynomials on \( B^{d-1} \) for the homogeneous polynomial
\[ \hat{\mathcal{P}}(x') := \sum_{\|x'\|_N = 0} (-1)^d a_q x'^1 \|x'\|^{2d}, \tag{22} \]
and for the homogeneous polynomial
\[ \hat{\mathcal{P}}(x') := \sum_{\|x'\|_{N-1} = 0} (-1)^d a_q x'^1 \|x'\|^{2d}, \tag{23} \]
respectively, with respect to the weight function \( \sqrt{1 - \|x'\|^2} \).

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Proposition 3.1. (a) The polynomial \( Q(x) \), even in \( x_d \), is a min–max polynomial on \( S^{d-1} \) if and only if \( \tilde{Q}(x') := Q(x', \sqrt{1 - ||x'||^2}) = Q(x', \sqrt{1 - ||x'||^2}) \) is a min–max polynomial on \( B^{d-1} \). Furthermore, \( ||Q||_{S^{d-1}} = ||Q||_{B^{d-1}} \).

(b) The polynomial \( Q(x) \), odd in \( x_d \), is a min–max polynomial on \( S^{d-1} \) if and only if \( \tilde{Q}(x') := Q(x', \sqrt{1 - ||x'||^2})/\sqrt{1 - ||x'||^2} = Q(x', \sqrt{1 - ||x'||^2})/\sqrt{1 - ||x'||^2} \) is a min–max polynomial on \( B^{d-1} \) with respect to the weight function \( \sqrt{1 - ||x'||^2} \). Furthermore, \( ||Q||_{S^{d-1}} = ||\tilde{Q}||_{B^{d-1}} \).

Proof. (a) We note first that for any polynomial \( P(x) \) even in \( x_d \),

\[
\max_{x \in S^{d-1}} |P(x)| = \max_{x \in S^{d-1}} |P(x)|,
\]

and therefore

\[
||P||_{S^{d-1}} = ||\tilde{P}||_{B^{d-1}}, \tag{24}
\]

where \( \tilde{P}(x') := P(x', \sqrt{1 - ||x'||^2}) \), and hence also for \( Q \) and \( \tilde{Q} \).

Assume first that \( Q \), even in \( x_d \), is a min–max polynomial on \( S^{d-1} \), that is, \( Q(x) = P(x) + q(x) \), where \( P \) is given by \( \Pi^d_{N-1} \) and \( q(x) \in \Pi^d_{N-1} \), even in \( x_d \). We want to prove that \( ||\tilde{P}||_{B^{d-1}} \geq ||Q||_{S^{d-1}} \) holds, for every polynomial \( \tilde{P} \in \Pi^d_{N-1} \) of the form of a min–max polynomial on \( B^{d-1} \), that is, of the form \( \tilde{P}(x') = \tilde{P}(x') + \tilde{p}(x') \), where \( \tilde{P} \) is defined by \( \Pi^d_{N-1} \) and \( \tilde{p} \) is such a polynomial and consider \( \tilde{P} \) to be such that \( P(x', \sqrt{1 - ||x'||^2}) = \tilde{P}(x') \). Clearly, \( P \) must be of the form \( P(x) = P(x) + p(x) \), where \( P \) is given by \( \Pi^d_{N-1} \) and \( p \in \Pi^d_{N-1} \), even in \( x_d \), and hence of the form of a min–max polynomial on \( S^{d-1} \). Since \( Q \) is min–max on \( S^{d-1} \), with the help of \( 24 \), it follows that

\[
||\tilde{P}||_{B^{d-1}} = ||P||_{S^{d-1}} \geq ||Q||_{S^{d-1}} = ||\tilde{Q}||_{B^{d-1}},
\]

and thus the first part of (a) is proved.

For the second part, assume \( \tilde{Q} \) is min–max on \( B^{d-1} \) and prove that for any \( P \in \Pi^d_{N} \) such that \( P(x) = P(x) + p(x) \), where \( P \) is given by \( \Pi^d_{N-1} \), \( ||p||_{S^{d-1}} \), which implies that \( Q \) is min–max on \( S^{d-1} \). We can assume \( P \) is even in \( x_d \); otherwise we can take the polynomial \( (P(x', x_d) + P(x', -x_d)) \). Hence \( \tilde{P}(x') := P(x', \sqrt{1 - ||x'||^2}) \) is of the form \( \tilde{P}(x') = \tilde{P}(x') + \tilde{p}(x') \), where \( \tilde{P} \) is given by \( \Pi^d_{N-1} \) that is, the form of a min–max polynomial on \( B^{d-1} \). With the help of \( 24 \) and the fact that \( \tilde{Q} \) is min–max on \( B^{d-1} \), we deduce that

\[
||P||_{S^{d-1}} = ||\tilde{P}||_{B^{d-1}} \geq ||\tilde{Q}||_{B^{d-1}} = ||Q||_{S^{d-1}},
\]

that is, \( Q \) is min–max on \( S^{d-1} \).

The proof of statement (b) is based on the fact that, for a polynomial \( P(x) \) odd in \( x_d \),

\[
||P||_{S^{d-1}} = ||\sqrt{1 - ||x'||^2} \tilde{P}||_{B^{d-1}}, \tag{25}
\]

where

\[
\tilde{P}(x') := P(x', \sqrt{1 - ||x'||^2})/\sqrt{1 - ||x'||^2}.
\]

The proof runs then along the same lines as that of (a). \( \square \)

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**Proof of Theorem 2.1.** (a) Let \( Q(x) \) denote the polynomial given by (7). We show first that \( Q \) has no extreme points in the interior of \( B^d \) by checking the solutions of the system

\[
\frac{\partial Q}{\partial x_1}(x) = 0, \quad \frac{\partial Q}{\partial x_2}(x) = 0, \ldots, \frac{\partial Q}{\partial x_d}(x) = 0.
\]

If \( \tilde{x} \) is one of its solutions, then by the first two equations, \( \frac{\partial P_{n_1}}{\partial x_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial P_{n_1}}{\partial x_2}(\tilde{x}_1, \tilde{x}_2) = 0 \) or \( P_{n_2}(\tilde{x}_3, \tilde{x}_4) \ldots P_{n(d-1)/2}(\tilde{x}_{d-2}, \tilde{x}_{d-1}) = 0 \) or \( q_m(\tilde{x}_d^2; (1-s)^{\lfloor n/2 \rfloor}) = 0 \). The condition \( \frac{\partial P_{n_1}}{\partial x_1}(\tilde{x}_1, \tilde{x}_2) = \frac{\partial P_{n_1}}{\partial x_2}(\tilde{x}_1, \tilde{x}_2) = 0 \) implies by Euler’s formula for homogeneous polynomials,

\[
x_1 \frac{\partial P_{n_1}}{\partial x_1}(x_1, x_2) + x_2 \frac{\partial P_{n_1}}{\partial x_2}(x_1, x_2) = n_1 P_{n_1}(x_1, x_2),
\]

that \( P_{n_1}(\tilde{x}_1, \tilde{x}_2) = 0 \). Therefore, if \( \tilde{x} \) is an extreme point of \( Q(x) \) in the interior of \( B^d \), then \( P_{n_1}(\tilde{x}_1, \tilde{x}_2) = 0 \) or \( P_{n_2}(\tilde{x}_3, \tilde{x}_4) \ldots P_{n(d-1)/2}(\tilde{x}_{d-2}, \tilde{x}_{d-1}) = 0 \) or \( q_m(\tilde{x}_d^2; (1-s)^{\lfloor n/2 \rfloor}) = 0 \), and hence \( Q(\tilde{x}) = 0 \). Thus \( Q \) attains its maximum modulus on the boundary \( S^{d-1} \) of \( B^d \) and for statement (a) to hold, it suffices to prove that \( Q \) is a min–max polynomial on \( S^{d-1} \). But since \( Q \) is even in \( x_d \), by Proposition 3.1(a), this is equivalent to

\[
\tilde{Q}(x') := Q(x', \sqrt{1 - \|x'\|^2})
\]

being a min–max polynomial on \( B^{d-1} \).

Since \( x' = (x_1, \ldots, x_{d-1}) \in B^{d-1} \) can be represented in the form \( (x_1, \ldots, x_{d-1}) = (r y_1, \ldots, r y_{d-1}) \), where \( r \in [0, 1] \), \( (y_1, \ldots, y_{d-1}) \in S^{d-2} \), by (27) and the fact that the polynomials \( P_{n_1}(x_1, x_2) \), \( P_{n_2}(x_3, x_4) \), \ldots, \( P_{n(d-1)/2}(x_{d-2}, x_{d-1}) \) are homogeneous, we have

\[
\tilde{Q}(r y_1, \ldots, r y_{d-1}) = (\prod_{k=1}^{(d-1)/2} P_{n_k}(y_{2k-1}, y_{2k})) r^{|n_1|} q_m(1 - r^2, (1-s)^{|n|}/2).
\]

To determine the maximum modulus of the polynomial \( \prod_{k=1}^{(d-1)/2} P_{n_k}(y_{2k-1}, y_{2k}) \) on \( S^{d-2} \), as well as its extreme points, we use the Lagrange multiplier rule. If \( (y_1, \ldots, y_{d-1}) \) is such an extreme point, then \( (y_{2k-1}, y_{2k}) \neq (0, 0) \), \( k = 1, \ldots, (d-1)/2 \); otherwise the polynomial would be zero. Without loss of generality, assume \( y_1, y_3, \ldots, y_{d-2} \neq 0 \). By straightforward calculation, the extreme points are among the solutions of the system

\[
\begin{align*}
y_{2k} \frac{\partial P_{n_k}(y_{2k-1}, y_{2k})}{\partial y_{2k-1}} - y_{2k-1} \frac{\partial P_{n_k}(y_{2k-1}, y_{2k})}{\partial y_{2k}} &= 0, \\
y_1 \frac{\partial P_{n_1}(y_1, y_2)}{\partial y_{2k-1}} - y_{2k-1} \frac{\partial P_{n_1}(y_1, y_2)}{\partial y_{2k}} &= y_{2k-1} \frac{\partial P_{n_k}(y_{2k-1}, y_{2k})}{\partial y_1}(y_1, y_2),
\end{align*}
\]

\( k = 1, 2, \ldots, (d-1)/2 \).

For any \( k = 1, 2, \ldots, (d-1)/2 \), by the first equation in (29) and by Euler’s formula (26) for \( P_{n_k}(y_{2k-1}, y_{2k}) \), we obtain that

\[
y_{2k-1} P_{n_k}(y_{2k-1}, y_{2k}) = \frac{\partial P_{n_k}(y_{2k-1}, y_{2k})}{\partial y_{2k-1}} (y_{2k-1}^2 + y_{2k}^2) \frac{y_{2k-1}^2 + y_{2k}^2}{n_k}.
\]

Using the relations (30), by the second equation in (29) for any \( k = 1, 2, \ldots, (d-1)/2 \), as well as the fact that the point \( (y_1, \ldots, y_{d-1}) \) is on the sphere \( S^{d-2} \), we reduce the system (29) to the
following:
\[
\begin{align*}
\frac{\partial P_{nk}}{\partial y_{2k-1}}(y_{2k-1}, y_{2k}) - \frac{\partial P_{nk}}{\partial y_{2k}}(y_{2k-1}, y_{2k}) &= 0 \\
y_{2k-1}^2 + y_{2k}^2 &= \frac{n_k}{|n|}, \quad k = 1, \ldots, (d - 1)/2.
\end{align*}
\] (31)

The solutions of this system are easy to find using the definition (5) of a homogeneous harmonic polynomial. They are \((y_1^{(j_1)}, y_2^{(j_2)}, y_3^{(j_3)}, y_4^{(j_2)}, \ldots, y_{d-2}^{(j_{(d-1)/2})}, y_{d-1}^{(j_{(d-1)/2})})\), with \(j_k = 0, 1, \ldots, 2n_k - 1, k = 1, 2, \ldots, (d - 1)/2\), where
\[
\begin{align*}
y_{2k-1}^{(j_k)} &= \sqrt{\frac{n_k}{|n|}} \cos \frac{\varphi_k + j_k \pi}{n_k}, \\
y_{2k}^{(j_k)} &= \sqrt{\frac{n_k}{|n|}} \sin \frac{\varphi_k + j_k \pi}{n_k},
\end{align*}
\] (32)

and \(\varphi_k \in [0, 2\pi)\) is uniquely defined by \(\cos \varphi_k = \alpha_k/\sqrt{\alpha_k^2 + \beta_k^2}\) and \(\sin \varphi_k = \beta_k/\sqrt{\alpha_k^2 + \beta_k^2}\).

Furthermore,
\[
\prod_{k=1}^{(d-1)/2} P_{nk}(y_{2k-1}, y_{2k}) = (-1)^{j_1 + j_2 + \cdots + j_{(d-1)/2}} \prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n, \tag{33}
\]
where \(M_n\) is defined by (6). Therefore,
\[
\left| \prod_{k=1}^{(d-1)/2} P_{nk}(y_{2k-1}, y_{2k}) \right| \leq \prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n \tag{34}
\]
holds for all \((y_1, \ldots, y_{d-1}) \in S^{d-2}\).

On the other hand, the transformation \(1 - r^2 = s\) (recall also (4)) yields
\[
|r^{|n|}q_m(1 - r^2; (1 - s)^{|n|}/2)| \leq E_{m-1}(s^m; (1 - s)^{|n|}/2) \tag{35}
\]
for all \(r \in [0, 1]\). Furthermore, by the Alternation Theorem there exist at least \(m + 1\) alternation points \(r_i \in (0, 1), i = 1, \ldots, m + 1,\) such that
\[
r_i^{|n|}q_m(1 - r^2; (1 - s)^{|n|}/2) = (-1)^i E_{m-1}(s^m; (1 - s)^{|n|}/2). \tag{36}
\]

Combining (28), (34) and (35), it follows that
\[
|\tilde{Q}(r_1, \ldots, r_{d-1})| \leq \prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; (1 - s)^{|n|}/2)
\]
for all \(r \in [0, 1]\) and \((y_1, \ldots, y_{d-1}) \in S^{d-2}\), and the maximum modulus is attained at the points
\[
\begin{pmatrix}
n_1 y_1^{(j_1)}, r_1 y_2^{(j_2)}, r_1 y_3^{(j_2)}, r_1 y_4^{(j_2)}, \ldots, r_1 y_{d-2}^{(j_{(d-1)/2})}, r_1 y_{d-1}^{(j_{(d-1)/2})} \\
n_2 y_1^{(j_1)}, r_2 y_2^{(j_2)}, r_2 y_3^{(j_2)}, r_2 y_4^{(j_2)}, \ldots, r_2 y_{d-2}^{(j_{(d-1)/2})}, r_2 y_{d-1}^{(j_{(d-1)/2})}
\end{pmatrix}, \tag{37}
\]
i = 1, 2, \ldots, m + 1, \; j_k = 0, 1, \ldots, 2n_k - 1, \; k = 1, 2, \ldots, (d - 1)/2;\) more precisely, by (33) and (36),
\[
|\tilde{Q}(r_1, \ldots, r_{d-1})| \leq \prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2} M_n E_{m-1}(s^m; (1 - s)^{|n|}/2)
\]
\[= (-1)^{i+j_1+j_2+\ldots+j_{(d-1)/2}} \prod_{k=1}^{(d-1)/2} \sqrt{\alpha_k^2 + \beta_k^2 M_{n-1}(s^m; (1-s)^{n/2})}.\]

The assertion (a) follows if we are able to prove that the points (37) with sign \((-1)^{i+j_1+j_2+\ldots+j_{(d-1)/2}}\) form the support of an extremal signature with respect to \(\Pi_{|n|+2m-1}^{d-1}\). For this, let us consider the d − 1 polynomials

\[p_k(x_1, \ldots, x_{d-1}) := \prod_{j_k=0}^{n_k-1} \left( \sin \frac{\varphi_k + j_k \pi}{n_k} x_{2k-1} - \cos \frac{\varphi_k + j_k \pi}{n_k} x_{2k} \right),\]

\[k = 1, 2, \ldots, (d - 1)/2,\]

\[p_{\frac{d-1}{2} + k}(x_1, \ldots, x_{d-1}) := x_{2k-1}^2 + x_{2k}^2 - \frac{n_k}{n_1} (x_1^2 + x_2^2), \quad k = 2, \ldots, (d - 1)/2,\]

\[p_{d-1}(x_1, \ldots, x_{d-1}) := \prod_{i=1}^{m+1} \left( x_1^2 + x_2^2 + \cdots + x_{d-1}^2 - r_i^2 \right).\]

The points (37) are precisely the common roots of the polynomials \(p_1, \ldots, p_{d-1}\). This can be easily seen since if \((x_1, \ldots, x_{d-1})\) is such a common root, then from the condition that it is root of \(p_1, \ldots, p_{(d-1)/2}\), its components must be of the form \(x_{2k-1} = c_k \cos(\varphi_k + j_k^\prime \pi)/n_k, x_{2k} = c_k \sin(\varphi_k + j_k^\prime \pi)/n_k, k = 1, \ldots, (d - 1)/2,\) for some \(j_k^\prime \in [0, 1, \ldots, n_k - 1]\) and some constant \(c_k\), and the constants are then all determined from the condition that \((x_1, \ldots, x_{d-1})\) is a root of \(p_{(d+1)/2}, \ldots, p_{d-1}\). By checking the sign of the Jacobian at the points (37), the above assertion concerning the extremal signature follows by Shapiro’s Theorem [12, Theorem 2], and (a) is thus proved.

As regards (b), after showing as in the proof of (a) that the polynomial given by (9) attains its maximum modulus on the boundary of \(B^d\), using Proposition 3.1(b), the problem is reduced to a weighted approximation problem on \(B^d\). The proof runs then along the same lines as that of (a). □

**Proof of Theorem 2.4.** The assertion follows by Theorem 2.1, with one of the homogeneous harmonic polynomials taken of degree 1, in conjunction with Remark 2.2. □

**Proof of Theorem 2.6.** Since \(\|a^\prime\| = 1\), at least one of \(a_1, \ldots, a_{d-1}\) is different from zero. Without loss of generality, assume \(a_1 \neq 0\) and therefore \(a_i^2 \neq 1, i = 2, 3, \ldots, d - 1\).

(a) Let \(Q(x)\) denote the polynomial from (a). As in the proof of Theorem 2.1(a) we can show that \(Q\) attains its maximum modulus on \(S^{d-1}\) and since \(Q\) is even in \(x_d\), by Proposition 3.1(a), the problem reduces to showing that

\[\hat{Q}(x^\prime) := Q(x^\prime, \sqrt{1 - \|x^\prime\|^2})\]

is a min–max polynomial on \(B^{d-1}\).

Using for \(x^\prime = (x_1, \ldots, x_{d-1}) \in B^{d-1}\) the representation \((x_1, \ldots, x_{d-1}) = (r y_1, \ldots, r y_{d-1})\), where \(r \in [0, 1], y^\prime = (y_1, \ldots, y_{d-1}) \in S^{d-2}\), we obtain by (38) that

\[\hat{Q}(r y_1, \ldots, r y_{d-1}) = T_n((a^\prime, y^\prime)) r^n q_m (1 - r^2; (1-s)^{n/2}).\]

(39)

For all \(y^\prime \in S^{d-2}\), by the Schwarz inequality and the assumption \(\|a^\prime\| = 1\), it holds that \(|\langle a^\prime, y^\prime \rangle| \leq 1\), and therefore

\[|T_n((a^\prime, y^\prime))| \leq 1.\]

(40)
Furthermore, the maximum modulus is attained at all the points \( y' \) situated on the intersection of the \( n + 1 \) parallel hyperplanes \( a_1x_1 + \cdots + a_{d-1}x_{d-1} = \cos \frac{\nu \pi}{n}, \nu = 0, 1, \ldots, n, \) with \( S^{d-2}. \) Intersecting this set further with any two-dimensional hyperplane containing the line passing through \((0, \ldots, 0)\) and \((a_1, \ldots, a_{d-1})\), we obtain exactly \( 2n \) points. We consider here the two-dimensional hyperplane described by the set of \( d - 3 \) equations

\[
a_2x_1 - a_1x_2 = 0, \quad a_3x_1 - a_1x_3 = 0, \ldots, a_{d-2}x_1 - a_1x_{d-2} = 0.
\]

Hence, \( T_n((a', y')) \) attains its maximum modulus at the solutions of the \( n + 1 \) systems of equations

\[
a_1x_1 + \cdots + a_{d-1}x_{d-1} = \cos \frac{\nu \pi}{n}
\]

\[
a_2x_1 - a_1x_2 = 0, \quad a_3x_1 - a_1x_3 = 0, \ldots, a_{d-2}x_1 - a_1x_{d-2} = 0
\]

\[
x_1^2 + x_2^2 + \cdots + x_{d-1}^2 = 1
\]

where \( \nu = 0, 1, \ldots, n. \) By straightforward calculation, we obtain that the solutions of the \( n + 1 \) systems are \((y_1^k, \ldots, y_{d-1}^k), (-y_1^k, \ldots, -y_{d-1}^k), k = 0, \ldots, n - 1, \) where

\[
y_1^k := a_1 \cos \frac{k\pi}{n} - \frac{a_1a_{d-1}}{\sqrt{1 - a_{d-1}^2}} \sin \frac{k\pi}{n}
\]

\[
y_2^k := \frac{a_2}{a_1}y_1^k, \quad y_3^k := \frac{a_3}{a_1}y_1^k, \ldots, y_{d-2}^k := \frac{a_{d-2}}{a_1}y_1^k
\]

\[
y_{d-1}^k := a_{d-1} \cos \frac{k\pi}{n} + \frac{a_{d-1}}{\sqrt{1 - a_{d-1}^2}} \sin \frac{k\pi}{n},
\]

where \( a_{d-1}^2 \neq 1, \) as remarked at the beginning of the proof. Thus we have, for \( k = 0, 1, \ldots, n - 1, \)

\[
T_n(a_1y_1^k + \cdots + a_{d-1}y_{d-1}^k) = (-1)^k
\]

\[
T_n(-a_1y_1^k - \cdots - a_{d-1}y_{d-1}^k) = (-1)^{n+k}.
\]

On the other hand, using the transformation \( 1 - r^2 = s \) (recall also (4)), it follows that

\[
|r^n q_m(1 - r^2; (1 - s)^{n/2})| \leq E_{m-1}(s^m; (1 - s)^{n/2})
\]

for all \( r \in [0, 1]. \) Moreover, by the Alternation Theorem, there exist at least \( m + 1 \) points \( r_i \in (0, 1], i = 1, \ldots, m + 1, \) at which \( r^n q_m(1 - r^2; (1 - s)^{n/2}) \) attains its maximum value with sign \((-1)^i)\).

Combining (39), (40) and (44) yields

\[
|\hat{Q}(r_y y_1, \ldots, r_y y_{d-1})| \leq E_{m-1}(s^m; (1 - s)^{n/2})
\]

for all \( r \in [0, 1], (y_1, \ldots, y_{d-1}) \in S^{d-2}, \) and the maximum modulus is attained at the points

\[
(r_i y_1^k, \ldots, r_i y_{d-1}^k), (-r_i y_1^k, \ldots, -r_i y_{d-1}^k), \ i = 1, \ldots, m + 1, k = 0, \ldots, n - 1.
\]

More precisely, for \( i = 1, \ldots, m + 1, k = 0, 1, \ldots, n - 1, \)

\[
\hat{Q}(r_i y_1^k, \ldots, r_i y_{d-1}^k) = (-1)^{i+k}
\]

\[
\hat{Q}(-r_i y_1^k, \ldots, -r_i y_{d-1}^k) = (-1)^{n+i+k}.
\]

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We show next that the points (46) with the signs given by (47) form the support of an extremal signature with respect to $\Pi_{n+2m-1}^{d-1}$, which completes the proof of part (a). For this we define the $d - 1$ polynomials
\begin{align*}
p_1(x_1, \ldots, x_{d-1}) &:= \prod_{i=1}^{m+1} (x_1^2 + x_2^2 + \cdots + x_{d-1}^2 - r_i^2) \\
p_2(x_1, \ldots, x_{d-1}) &:= \prod_{k=0}^{n-1} (\alpha_k x_1 + \cdots + \alpha_{d-1} x_{d-1}) \\
p_3(x_1, \ldots, x_{d-1}) &:= a_2 x_1 - a_1 x_2 \\
\vdots \\
p_{d-1}(x_1, \ldots, x_{d-1}) &:= a_{d-2} x_1 - a_1 x_{d-1},
\end{align*}
where for any $k = 0, \ldots, n - 1$, $(\alpha_k^1, \ldots, \alpha_{d-1}^k) \in S^{d-2}$ and $\alpha_1^k x_1 + \cdots + \alpha_{d-1}^k x_{d-1} = 0$ is the hyperplane which intersects the two-dimensional hyperplane described by (41) along the line passing through the points $(0, \ldots, 0), (r_i^1, \ldots, r_i^{d-1})$ and $(-r_i^1, \ldots, -r_i^{d-1}), i = 1, \ldots, m + 1$. Hence the points (46) are the only common roots of the polynomials $p_1, \ldots, p_{d-1}$.
Checking the sign of the Jacobian at these points, by Shapiro’s Theorem [12, Theorem 2], the assertion about the extremal signature follows and (a) is thus proved.

The proof of part (b) is similar to that of (a), except that in this case Proposition 3.1(b) is used instead. □

**Proof of Corollary 2.8.** The statement of the corollary follows immediately by Theorem 2.1 and Proposition 3.1. □

**Proof of Corollary 2.9.** The statement follows by Theorem 2.1(a) with $n = 2$, taking into consideration representation (5) of a homogeneous harmonic polynomial and the fact that the approximation problem on $[0, 1]$ with respect to weight function $w(s) = 1 - s$ can be rewritten as an approximation problem on $[-1, 1]$ with respect to the weight function $w(t) = 1 - t^2$, for which the min–max polynomial can be written down explicitly; see the explanation before the statement of the corollary. □

**Proof of Corollary 2.10.** The statement of the corollary is precisely that of Corollary 2.8 with $d = 3$. □

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