On Complex (Non-Analytic) Chebyshev Polynomials in $\mathbb{C}^2$

Ionela Moale and Peter Yuditskii

(Communicated by Alexandre Eremenko)

Dedicated to the memory of Franz Peherstorfer

Abstract. We consider the problem of finding a best uniform approximation to the standard monomial on the unit ball in $\mathbb{C}^2$ by polynomials of lower degree with complex coefficients. We reduce the problem to a one-dimensional weighted minimization problem on an interval. In a sense, the corresponding extremal polynomials are uniform counterparts of the classical orthogonal Jacobi polynomials. They can be represented by means of special conformal mappings on the so-called comb-like domains. In these terms, the value of the minimal deviation and the representation for a polynomial of best approximation for the original problem are given. Furthermore, we derive asymptotics for the minimal deviation.

Keywords. Polynomial approximation, uniform norm, several variables, minimal deviation, asymptotics, conformal mapping.


1. Introduction

We consider the standard basis in the set of (non-analytic) complex polynomials in $\mathbb{C}^2$:

$$\left\{ z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} \right\}_{k_1 \geq 0, l_1 \geq 0, k_2 \geq 0, l_2 \geq 0}, \quad (z_1, z_2) \in \mathbb{C}^2.$$

As usual $k_1 + l_1 + k_2 + l_2$ is called the total degree of the given monomial.

In what follows we use the following notation: $\Pi_n$ denotes the set of polynomials with complex coefficients of total degree less than or equal to $n$, and $\|P\|$ denotes

Received February 19, 2010, in revised form March 17, 2010.
Published online May 20, 2010.
The first author was supported by the Austrian Science Fund FWF, project no: P20413-N18.
The second author was supported by the Austrian Science Fund FWF, project no: P22025-N18.

ISSN 1617-9447/2 2.50 © 2011 Heldermann Verlag
the uniform norm of $P \in \Pi_n$ in the complex ball

$$\|P\| = \sup_{(z_1, z_2) \in B} |P(z_1, z_2)|, \quad B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 \leq 1\}.$$ 

Analogously to the classical Chebyshev polynomial, we consider the best approximation on the ball $B$ of the monomial $z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2}$ by polynomials of total degree less than $n := k_1 + l_1 + k_2 + l_2$. Such a polynomial

$$\tilde{T}_{k_1, l_1, k_2, l_2}(z_1, z_2) = z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} + \ldots,$$

which we call a polynomial of least deviation from zero on $B$, is not unique but the minimal deviation $L_{k_1, l_1, k_2, l_2}$ is well defined,

$$L_{k_1, l_1, k_2, l_2} := \inf_{P \in \Pi_{n-1}} \|z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} - P(z_1, z_2)\|.$$ 

It is convenient to work with the normalized polynomial

$$T_{k_1, l_1, k_2, l_2} := \frac{\tilde{T}_{k_1, l_1, k_2, l_2}}{\|\tilde{T}_{k_1, l_1, k_2, l_2}\|}.$$ 

Thus

$$T_{k_1, l_1, k_2, l_2}(z_1, z_2) = \Lambda_{k_1, l_1, k_2, l_2} z_1^{k_1} \bar{z}_1^{l_1} z_2^{k_2} \bar{z}_2^{l_2} + \ldots,$$

where $\Lambda_{k_1, l_1, k_2, l_2} = 1/L_{k_1, l_1, k_2, l_2}$.

Concerning polynomials of least deviation from zero to monomials on the unit ball in $\mathbb{R}^2$, several approaches are known so far, see [4, 7, 13] and also [11]. Of foremost importance to us was the representation of the extremal polynomial given by Braß in [4]. Here we essentially simplify and generalize his construction.

**Remark 1.** More precisely, Braß [4] considered the region

$$Q := \{(x_1, \ldots, x_d) : -1 \leq x_1 \leq x_2 \leq \ldots \leq x_d \leq 1\}, \quad d \geq 1,$$

and gave a solution to the problem of determining the best uniform approximation to the multivariate monomials, see [4, Satz 1]. As it was pointed out by the author, for $d = 2$ there is a connection between the region $Q$ and the unit ball. Therefore in this case, new polynomials of least deviation from zero also on the unit ball are obtained, but it was not mentioned there and we are also unable to see such a connection between the corresponding region and the unit ball for $d \geq 3$. In our study we arrive at a weighted analogue of this problem, see Section 2 and the proof of Theorems 1 and 3 below. In this case, the technique that was used in the cited paper appears to be redundant, since it uses too many explicit properties of the classical Chebyshev polynomials.

**Remark 2.** In [13], Reimer defined by means of a generating function a class of polynomials in any dimension $d \geq 1$, and proved that they are polynomials of least deviation from zero to monomials for $d \in \{1, 2\}$, see [13, Thm. 2].
In approximation theory, a special role of the conformal mappings on so-called comb-like domains is well known, see e.g. the book [1], the survey [14] and the references on original papers therein, in particular [2, 9]. For recent developments in this direction, see [6, 12]. By analogy with the MacLane-Vinberg special representation for polynomials and entire functions [10, 15], in this paper to non-negative real numbers $\alpha, \beta$ and an integer $n \geq 0$ we associate a horizontal strip with $n + 1$ horizontal cuts, see Figure 1:

$$\Omega_n(\alpha, \beta) = \left\{ w = u + iv: -\beta < \frac{v}{\pi} < \alpha + n \right\} \setminus \bigcup_{j=0}^{n} \left\{ w = u + iv: \frac{v}{\pi} = j, u \leq 0 \right\}.$$ 

![Figure 1. The domain $\Omega_2(\alpha, \beta)$.](image)

Note that the boundary of the domain contains $n + 3$ infinite points:

$$\begin{align*}
\infty_0 &= -\infty + iv, n < \frac{v}{\pi} < n + \alpha, \\
\infty_j &= -\infty + iv, n - j < \frac{v}{\pi} < n - j + 1, \\
\infty_{n+1} &= -\infty + iv, -\beta < \frac{v}{\pi} < 0, \\
\infty_{n+2} &= +\infty + iv, -\beta < \frac{v}{\pi} < \alpha + n.
\end{align*}$$

Let $w: \mathbb{C}_+ \to \Omega_n(\alpha, \beta)$ be the conformal mapping of the upper half-plane onto $\Omega_n(\alpha, \beta)$ with the following normalization

$$w(0) = \infty_0, \quad w(1) = \infty_{n+1}, \quad w(\infty) = \infty_{n+2}.$$ 

It is easy to see that it has the following asymptotics at infinity ($z \to \infty$)

\begin{equation}
(1) \quad w(z) = w_n(z; \alpha, \beta) = (\alpha + \beta + n) \ln z + C_n(\alpha, \beta) - \beta \pi i + O\left(\frac{1}{z}\right)
\end{equation}

The real constant $C_n(\alpha, \beta)$ is uniquely defined by the domain (a kind of capacity).
Due to the evident symmetry
\[
\Lambda_{k_1,l_1,k_2,l_2} = \Lambda_{l_1,k_1,k_2,l_2} = \Lambda_{k_1,l_1,k_2,l_2}
\]
we can assume that \( k_1 \geq l_1 \) and \( k_2 \geq l_2 \). Our first result is

**Theorem 1.** With the above introduced notation
\[
\ln \Lambda_{k_1,l_1,k_2,l_2} = C_{l_1+l_2} \left( \frac{k_1 - l_1}{2}, \frac{k_2 - l_2}{2} \right).
\]

Below we give the representation for a polynomial \( T_{k_1,l_1,k_2,l_2}(z_1, z_2) \) of least deviation from zero. For this, we establish a connection between the conformal mapping \( w_n(z; \alpha, \beta) \) and a weighted 1-D extremal problem on \([0, 1]\). In a sense, in the following proposition we define *uniform Jacobi polynomials*, to compare to the classical orthogonal ones \([3]\).

**Proposition 2.** Let \( \xi_j = w_n^{-1}(\infty; \alpha, \beta), \ 1 \leq j \leq n \). Then
\[
e^{w_n(t;\alpha,\beta)} = t^\alpha (1-t)^{\beta} e^{C_n(\alpha,\beta)}(t - \xi_1) \cdots (t - \xi_n),
\]
where
\[
\tilde{J}_n(t; \alpha, \beta) := (t - \xi_1) \cdots (t - \xi_n) = t^n + \cdots
\]
is the polynomial of least deviation from zero on \([0, 1]\) with respect to the weight \( t^\alpha (1-t)^{\beta} \). Moreover
\[
\| \tilde{J}_n(t; \alpha, \beta) \| = \sup_{0 \leq t \leq 1} t^\alpha (1-t)^{\beta} |\tilde{J}_n(t; \alpha, \beta)| = e^{-C_n(\alpha,\beta)},
\]
that is,
\[
e^{w_n(t;\alpha,\beta)} = t^\alpha (1-t)^{\beta} \tilde{J}_n(t; \alpha, \beta),
\]
as before
\[
\tilde{J}_n(t; \alpha, \beta) : = \frac{\tilde{J}_n(t; \alpha, \beta)}{\| \tilde{J}_n(t; \alpha, \beta) \|},
\]
We point out that \( \xi_l < \xi_{l+1} \) for all \( 1 \leq l \leq n - 1 \).

**Theorem 3.** Let \( \alpha = (k_1 - l_1)/2 \geq 0 \) and \( \beta = (k_2 - l_2)/2 \geq 0 \). Let us factorize
\[
\tilde{J}_{l_1+l_2}(t; \alpha, \beta) = \tilde{J}_{l_1}^{(1)}(t) \tilde{J}_{l_2}^{(2)}(t)
\]
in polynomials of degrees \( l_1 \) and \( l_2 \) respectively in the following way
\[
\tilde{J}_{l_1}^{(1)}(t) = (t - \xi_1) \cdots (t - \xi_{l_1}),
\]
\[
\tilde{J}_{l_2}^{(2)}(t) = (t - \xi_{l_1+1}) \cdots (t - \xi_{l_1+l_2}).
\]

Then
\[
T_{k_1,l_1,k_2,l_2}(z_1, z_2) = e^{C_{l_1+l_2}(\alpha,\beta)} z_1^{k_1-l_1} z_2^{k_2-l_2} \tilde{J}_{l_1}^{(1)}(|z_1|^2) (-1)^{l_2} \tilde{J}_{l_2}^{(2)}(1 - |z_2|^2)
\]
Finally we present the following asymptotic relation for the value of the minimal deviation.
Theorem 4. Assume that the following limits exist
\[
\kappa_1 = \lim_{n \to \infty} \frac{k_1}{n}, \quad \lambda_1 = \lim_{n \to \infty} \frac{l_1}{n}, \quad \kappa_2 = \lim_{n \to \infty} \frac{k_2}{n}, \quad \lambda_2 = \lim_{n \to \infty} \frac{l_2}{n},
\]
where \( n = k_1 + l_1 + k_2 + l_2 \). Then
\[
\lim_{n \to \infty} L_{k_1,l_1,k_2,l_2}^{2/n} = (\lambda_1 + \lambda_2)\kappa_1 + \lambda_2 (\kappa_1 + \kappa_2)\kappa_1 \lambda_2 (\kappa_1 + \kappa_2)^{\kappa_1 + \lambda_2}.
\]

2. Reduction to 1-D problem

First, we reduce our complex two-dimensional approximation problem to a weighted approximation problem in two real variables on the standard triangle
\[
\Delta := \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \geq 0, t_2 \geq 0, t_1 + t_2 \leq 1\}.
\]
For a continuous function \( f \) on \( \Delta \), we define
\[
\|f\|_\Delta := \max_{(t_1, t_2) \in \Delta} |f(t_1, t_2)|.
\]
By
\[
Y_{l_1,l_2}(t_1, t_2; \alpha, \beta) = M_{l_1,l_2}(\alpha, \beta) t_1^{l_1} t_2^{l_2} + \cdots
\]
we denote a normalized polynomial of least deviation from zero on \( \Delta \) with respect to the weight function \( t_1^\alpha t_2^\beta \).

Proposition 5. Let \( \alpha = (k_1 - l_1)/2 \) and \( \beta = (k_2 - l_2)/2 \). Then
\[
z_1^{k_1-l_1} z_2^{k_2-l_2} Y_{l_1,l_2}(|z_1|^2, |z_2|^2; \alpha, \beta)
\]
is a normalized polynomial of least deviation from zero on \( \mathbb{B} \), that is,
\[
\Lambda_{k_1,l_1,k_2,l_2} = M_{l_1,l_2}(\alpha, \beta).
\]

Proof. Let us remark that due to the symmetries of \( \mathbb{B} \), if \( T_{k_1,l_1,k_2,l_2}(z_1, z_2) \) is a polynomial of least deviation from zero, then for any \( \theta_1, \theta_2 \in [0, 2\pi] \), the polynomials
\[
T_{k_1,l_1,k_2,l_2}(e^{i\theta_1} z_1, e^{i\theta_2} z_2) e^{-i(k_1-l_1)\theta_1} e^{-i(k_2-l_2)\theta_2}
\]
and
\[
\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} T_{k_1,l_1,k_2,l_2}(e^{i\theta_1} z_1, e^{i\theta_2} z_2) e^{-i(k_1-l_1)\theta_1} e^{-i(k_2-l_2)\theta_2} d\theta_1 d\theta_2
\]
are also polynomials of least deviation from zero.

It is easy to see that the polynomial in (6) is of the form
\[
\Lambda_{k_1,l_1,k_2,l_2} z_1^{k_1-l_1} z_2^{k_2-l_2} \left( |z_1|^{2l_1} |z_2|^{2l_2} + \sum_{c+j+l_1+l_2-1} a_{j_1,j_2} |z_1|^{2j_1} |z_2|^{2j_2} \right)
\]
\[
= \Lambda_{k_1,l_1,k_2,l_2} z_1^{k_1-l_1} z_2^{k_2-l_2} \hat{P}_{l_1,l_2} (|z_1|^2, |z_2|^2)
\]
where \( a_{j_1,j_2} \in \mathbb{C} \). Note that
\[
\hat{T}_{k_1,l_1,k_2,l_2}(z_1, z_2) := z_1^{k_1-l_1} z_2^{k_2-l_2} Q_{l_1,l_2} (|z_1|^2, |z_2|^2),
\]
where
\[ Q_{l_1,l_2}(\|z_1\|^2, \|z_2\|^2) := \Lambda_{k_1,l_1,k_2,l_2} \text{Re} \tilde{P}_{l_1,l_2}(\|z_1\|^2, \|z_2\|^2), \]
is still a normalized polynomial of least deviation from zero on \( \mathbb{B} \).
Since \((z_1, z_2) \in \mathbb{B}\) is equivalent to \((t_1, t_2) \in \Delta\), where \(t_1 := \|z_1\|^2, t_2 := \|z_2\|^2\), we have that
\[ \|T_{k_1,l_1,k_2,l_2}\|_\mathbb{B} = \|t_1^{(k_1-l_1)/2}t_2^{(k_2-l_2)/2}Q_{l_1,l_2}\|_\Delta, \]
which gives the assertion. \( \blacksquare \)

Now we give a sufficient condition for a polynomial to be a weighted polynomial of least deviation from zero on \( \Delta \). In the next section we show the existence of a polynomial satisfying this condition.

As before \( J_n(t; \alpha, \beta) \) denotes the normalized polynomial of least deviation from zero on \([0, 1]\) with respect to the weight function \( t^\alpha (1-t)^\beta \).

**Proposition 6.** Let \( \alpha, \beta \geq 0 \) and let \( c_n = c_n(\alpha, \beta) > 0 \) be the leading coefficient of \( J_n(t; \alpha, \beta) \), \( J_n(t; \alpha, \beta) = c_nt^n + \cdots \). If there exists a polynomial
\[ (7) \quad P_{l_1,l_2}(t_1,t_2) = c_{l_1+l_2}(\alpha, \beta)t_1^{l_1}t_2^{l_2} + \cdots \quad \text{with } \|t_1^{\alpha}t_2^{\beta}P_{l_1,l_2}\|_\Delta \leq 1 \]
such that \( P_{l_1,l_2}(t, 1-t) = (-1)^{l_2}J_{l_1+l_2}(t; \alpha, \beta) \) for all \( t \in [0, 1] \), then
\[ (8) \quad M_{l_1,l_2}(\alpha, \beta) = c_{l_1+l_2}(\alpha, \beta). \]
That is, the given \( P_{l_1,l_2} \) is a normalized polynomial of least deviation from zero on \( \Delta \) with respect to the weight \( t_1^{\alpha}t_2^{\beta} \).

**Proof.** Actually, we have to prove (8).

By the fact that \( Y_{l_1,l_2} \) is a polynomial of least deviation from zero, from (7) we have immediately that \( M_{l_1,l_2} \geq c_{l_1+l_2} \).

On the other hand, let us restrict \( Y_{l_1,l_2} \) to the line \( t_1 = t, t_2 = 1-t \):
\[ Q(t) := (-1)^{l_2}Y_{l_1,l_2}(t, 1-t) = M_{l_1,l_2}t^{l_1+l_2} + \cdots \]
Since \( |Q(t)t^\alpha (1-t)^\beta| \leq 1 \) for all \( t \in [0, 1] \), the extremal property of \( J_{l_1+l_2} \) implies \( M_{l_1,l_2} \leq c_{l_1+l_2} \). Thus the statement is proved. \( \blacksquare \)

### 3. Proofs of Proposition 2, Theorems 1 and 3

**Proof of Proposition 2** Let \( \eta_k = w_n^{-1}((n-k)\pi i; \alpha, \beta), 0 \leq k \leq n \). From the Schwarz-Christoffel formula, see e.g. [5], we obtain the following expression for the differential of the conformal mapping \( w_n(z; \alpha, \beta) \):
\[ (9) \quad dw_n(z; \alpha, \beta) = C\frac{\prod_{k=0}^{n}(z - \eta_k)}{z(z-1)\prod_{j=1}^{n}(z - \xi_j)} \, dz, \]
where $C \in \mathbb{C}$ is a constant. Having in mind the asymptotic behavior at the infinite boundary points of the domain $\Omega_n(\alpha, \beta)$ we get the following expansion into partial fraction for (9):

$$dw_n(z; \alpha, \beta) = \left(\frac{\alpha}{z} + \frac{\beta}{z-1} + \sum_{j=1}^{n} \frac{1}{z - \xi_j}\right) dz.$$ 

Hence

$$w_n(z; \alpha, \beta) = \alpha \ln z + \beta \ln(z - 1) + \sum_{j=1}^{n} \ln(z - \xi_j) + C_1,$$

where $C_1 \in \mathbb{C}$ is a constant. Relation (2) follows now immediately from (10).

From the boundary correspondence for the given conformal mapping we get that the function $t^\alpha(1-t)^\beta J_n(t; \alpha, \beta)$ alternates $n + 1$ times between $\pm1$ on $[0,1]$, see Figure 2. Thus the Chebyshev alternation theorem implies that $J_n(t; \alpha, \beta)$ is indeed the polynomial of least deviation from zero with respect to the given weight with leading coefficient $c_n(\alpha, \beta) = e^{C_n(\alpha, \beta)}$.

**Figure 2.** Graph of $t^\alpha(1-t)^\beta J_2(t; \alpha, \beta)$, $\alpha = 1/2$, $\beta = 2$.

**Proof of Theorems 1 and 3.** By Proposition 5, the assertion of Theorem 3 follows if we are able to show that

$$P_{l_1,l_2}(t_1, t_2; \alpha, \beta) := e^{C_{l_1+l_2}(\alpha, \beta)} J_{l_1}^{(1)}(t_1)(-1)^{l_2} J_{l_2}^{(2)}(1-t_2)$$

$$= e^{C_{l_1+l_2}(\alpha, \beta)} t_1^{l_1} t_2^{l_2} + \ldots$$
is a normalized polynomial of least deviation from zero on $\Delta$ with respect to the weight $t_1^\alpha t_2^\beta$, for which we will use Proposition 6.

By restricting the polynomial $P_{t_1,t_2}$ to the line $t_1 := t$, $t_2 := 1 - t$, we obtain that for all $t \in [0,1]$:
\begin{equation}
P_{t_1,t_2}(t,1-t;\alpha,\beta) = (-1)^{l_2}J_{t_1+l_2}(t;\alpha,\beta).
\end{equation}

Thus it remains to show that $\|t_1^\alpha t_2^\beta P_{t_1,t_2}\|_{\Delta} \leq 1$.

Let
\begin{equation}
F(t_1,t_2;\alpha,\beta) := t_1^\alpha t_2^\beta P_{t_1,t_2}(t_1,t_2;\alpha,\beta).
\end{equation}

Clearly $F(t_1,t_2;\alpha,\beta)$ is a product of two univariate functions, see (11). We normalize the first factor $f_1(t_1)$ by the condition $f_1(\eta_1) = 1$. Due to the definition of $\eta_1$ we have
\begin{equation}
F(\eta_1, 1-\eta_1;\alpha,\beta) = (-1)^{l_2}e^{w_\alpha(\eta_1;\alpha,\beta)} = 1.
\end{equation}

Thus
\begin{equation}
F(t_1,t_2;\alpha,\beta) = f_1(t_1)f_2(t_2), \quad f_2(1-\eta_1) = 1 (= f_1(\eta_1)).
\end{equation}

In addition, since $\xi_1 < \eta_1$, we can easily check, see (4) and (12), that $f_1(t)$ is strictly increasing for $t \in [\eta_1,1]$, in particular $f_1(t) \geq f_1(\eta_1) = 1$ here. Since $1-\xi_1+1 < 1-\eta_1$, $f_2(t)$ is strictly increasing and $f_2(t) \geq 1$ for $t \in [1-\eta_1,1]$.

We note that by (12) and (3):
\begin{equation}
|F(t,1-t;\alpha,\beta)| = t^\alpha(1-t)^\beta|J_{t_1+l_2}(t;\alpha,\beta)| \leq 1,
\end{equation}
for all $t \in [0,1]$. In order to show the main claim
\begin{equation}
|F(t_1,t_2;\alpha,\beta)| \leq 1
\end{equation}
for all $(t_1,t_2) \in \Delta$, we distinguish three regions in $\Delta$.

If $t \in [0,\eta_1]$ then (13) yields $|f_1(t)| \leq 1/|f_2(1-t)| \leq 1$, where the last inequality follows by the above listed properties of $f_2$. Similarly we obtain $|f_2(t)| \leq 1$, if $t \in [0,1-\eta_1]$. Thus
\begin{equation}
|F(t_1,t_2;\alpha,\beta)| = |f_1(t_1)f_2(t_2)| \leq 1 \quad \text{for } t_1 \in [0,\eta_1], t_2 \in [0,1-\eta_1].
\end{equation}

If $\eta_1 \leq t_1 \leq 1 - t_2 \leq 1$, then since $f_1$ is increasing on $[\eta_1,1]$, it follows that
\begin{equation}
|f_1(t_1)f_2(t_2)| \leq |f_1(t_1)f_2(1-t_2)| = |F(1-t_2,t_2;\alpha,\beta)|,
\end{equation}

hence (14) follows by (13).

If $1-\eta_1 \leq t_2 \leq 1 - t_1 \leq 1$, then since $f_2$ is increasing on $[1-\eta_1,1]$, it follows that
\begin{equation}
|f_1(t_1)f_2(t_2)| \leq |f_1(t_1)f_2(1-t_1)| = |F(t_1,1-t_1;\alpha,\beta)|,
\end{equation}

hence (14) follows again by (13).

By combining the three cases it follows that relation (14) holds for all $(t_1,t_2) \in \Delta$. In conclusion, relations (12) and (14) being proved, by Proposition 6 it follows that $P_{t_1,t_2}(t_1,t_2;\alpha,\beta)$ is a normalized polynomial of least deviation from zero on $\Delta$. 


with respect to the weight $t_1\alpha t_2\beta$, and hence, the polynomial given by (5) is a
polynomial of least deviation from zero on $\mathbb{B}$, which also proves Theorem 1. ■

4. Leading term in asymptotics

We need certain properties of the conformal mapping $w_*$ of the upper half-plane
onto the domain

$$\Omega_* = \{w = u + iv: -\beta\pi < v < (1 - \beta)\pi\} \setminus \{w = u + iv: u \leq 0, 0 \leq v \leq \alpha\pi\},$$

see Figure 3. Due to the Schwarz-Christoffel formula [5], it is of the form

$$w_*(z; \alpha, \beta) = \int_{x_2}^{z} \frac{\sqrt{(z - x_1)(z - x_2)}}{z(z - 1)} dz$$

where $x_1, x_2, 0 < x_1 < x_2 < 1$, are the preimages of the angle-points $\pi\alpha i$ and $0$
respectively. As before three “infinite points” in the domain correspond to $0, 1$
and $\infty$ and due to the size of corresponding strips we have the following relations

(15) \[ \alpha = \sqrt{x_1 x_2}, \quad \beta = \sqrt{(1 - x_1)(1 - x_2)}. \]

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \draw (-2,0) -- (2,0) node[right] {$\Omega_*$};
  \draw (0,-2) -- (0,2) node[above] {$(1 - \beta)\pi i$};
  \draw (0,0) -- (0,2) node[above] {$\alpha\pi i$};
  \draw (0,0) -- (0,-2) node[below] {$0$};
  \draw (0,0) -- (-1,-1) node[below] {$-\beta\pi i$};
\end{tikzpicture}
\caption{The domain $\Omega_*(\alpha, \beta)$.}
\end{figure}
This is an elementary integral, so we get

\begin{equation}
\begin{aligned}
w^*(z; \alpha, \beta) &= \sqrt{x_1 x_2} \ln \left( \frac{\sqrt{x_2 (1 - \frac{x_1}{x_2})} - \sqrt{x_1 (1 - \frac{x_2}{x_1})}}{x_2 - x_1} \right) \\
&+ \sqrt{(1 - x_1)(1 - x_2)} x_1 x_2 \\
&\times \ln \left( \frac{(1 - x_2)(1 - \frac{1 - x_1}{1 - z}) - (1 - x_1)(1 - \frac{1 - x_2}{1 - z})}{x_1 - x_2} \right) \\
&+ \frac{\ln \left( \frac{\sqrt{z - x_1} + \sqrt{z - x_2}}{x_2 - x_1} \right)^2}{x_2 - x_1}.
\end{aligned}
\end{equation}

Similarly to (1) we define the real constant $C^*(\alpha, \beta)$ by the condition

\begin{equation}
w^*(z; \alpha, \beta) = \ln z + C^*(\alpha, \beta) - \beta \pi i + O\left(\frac{1}{z}\right), \quad z \to \infty.
\end{equation}

By (16) we get

\begin{equation}
\begin{aligned}
C^*(\alpha, \beta) &= \sqrt{x_1 x_2} \ln \left( \frac{\sqrt{x_2} - \sqrt{x_1}}{x_2 - x_1} \right) \\
&+ \sqrt{(1 - x_1)(1 - x_2)} \ln \left( \frac{\sqrt{1 - x_2} - \sqrt{1 - x_1}}{x_2 - x_1} \right) \\
&+ \frac{4}{x_2 - x_1},
\end{aligned}
\end{equation}

which we simplify to

\begin{equation}
\begin{aligned}
C^*(\alpha, \beta) &= (1 - \sqrt{x_1 x_2} - \sqrt{(1 - x_1)(1 - x_2)}) \ln \frac{4}{x_2 - x_1} \\
&+ 2 \sqrt{x_1 x_2} \ln \frac{2}{\sqrt{x_2} + \sqrt{x_1}} \\
&+ 2 (1 - x_1)(1 - x_2) \ln \frac{2}{\sqrt{1 - x_2} + \sqrt{1 - x_1}}.
\end{aligned}
\end{equation}

Using (15) we get

\begin{equation}
\begin{aligned}
C^*(\alpha, \beta) &= \frac{1 - \alpha - \beta}{2} \ln \frac{16}{(1 - (\alpha + \beta)^2)(1 - (\alpha - \beta)^2)} \\
&+ \alpha \ln \frac{4}{(1 + \alpha)^2 - \beta^2} + \beta \ln \frac{4}{(1 + \beta)^2 - \alpha^2}.
\end{aligned}
\end{equation}

**Proof of Theorem 4.** As the sequence of domains

\begin{equation}
\frac{2}{n} \Omega_{n+1} \left( \frac{k_1 - l_1}{2}, \frac{k_2 - l_2}{2} \right)
\end{equation}
converges to \( \Omega^* \) as \( n \to \infty \), it follows by Carathéodory’s Theorem, see e.g. [8], that for the sequence of conformal mappings it holds that
\[
 w_*(z; \kappa_1 - \lambda_1, \kappa_2 - \lambda_2) = \lim_{n \to \infty} \frac{2}{n} w_{l_1+l_2} \left( z; \frac{k_1 - l_1}{2}, \frac{k_2 - l_2}{2} \right). 
\]

Therefore
\[
 \lim_{n \to \infty} \frac{2}{n} \ln \Lambda_{k_1,l_1,k_2,l_2} = \lim_{n \to \infty} \frac{2}{n} C_{l_1+l_2} \left( \frac{k_1 - l_1}{2}, \frac{k_2 - l_2}{2} \right) = C_*(\kappa_1 - \lambda_1, \kappa_2 - \lambda_2).
\]

Since in this case
\[
1 - \alpha - \beta = 2(\lambda_1 + \lambda_2) \\
1 + \alpha + \beta = 2(\kappa_1 + \kappa_2) \\
1 - \alpha + \beta = 2(\lambda_1 + \kappa_2) \\
1 + \alpha - \beta = 2(\kappa_1 + \lambda_2)
\]
by (17) we get
\[
C_*(\kappa_1 - \lambda_1, \kappa_2 - \lambda_2) = -(\kappa_1 + \kappa_2) \ln(\kappa_1 + \kappa_2) \\
-(\kappa_1 + \lambda_2) \ln(\kappa_1 + \lambda_2) \\
-(\kappa_1 + \lambda_2) \ln(\kappa_1 + \lambda_2) \\
-(\lambda_1 + \kappa_2) \ln(\lambda_1 + \kappa_2).
\]

References


---

**Ionela Moale**

E-MAIL: Ionela.Moale@jku.at

ADDRESS: Johannes Kepler University, Institute for Analysis, A-4040 Linz, Austria.

**Peter Yuditskii**

E-MAIL: Petro.Yudytskiy@jku.at

ADDRESS: Johannes Kepler University, Institute for Analysis, A-4040 Linz, Austria.